

THE CALCULUS FOR
ENGINEERS

BY EWART S. ANDREWS, B.Sc.ENG. (LOND.).

ELEMENTARY PRINCIPLES OF
REINFORCED CONCRETE
CONSTRUCTION.

BEING VOLUME I. OF
THE BROADWAY ENGINEERING HANDBOOKS.

THIRD REVISED EDITION.

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PUBLISHED BY

SCOTT, GREENWOOD & SON
8 BROADWAY, LUDGATE HILL, LONDON, E.C.4.

THE BROADWAY ENGINEERING HANDBOOKS
VOLUME XIII

THE CALCULUS FOR ENGINEERS

BY

EWART S. ANDREWS, B.Sc. ENG. (Lond.)

AUTHOR OF "THEORY AND DESIGN OF STRUCTURES"
FELLOW OF THE CHARTERED INSTITUTE OF PATENT AGENTS
FORMERLY DEMONSTRATOR AND LECTURER IN THE
ENGINEERING DEPARTMENT OF UNIVERSITY
COLLEGE, LONDON

AND

H. BRYON HEYWOOD, D.Sc. (PARIS),
B.Sc. (LOND.)

SECOND REVISED EDITION

*With One Hundred and Two Illustrations, Tables, and
Numerous Worked Examples*



P.O. Bal

LONDON

SCOTT, GREENWOOD & SON

28 BROADWAY, LUDGATE HILL, E.C. 4

1922

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PRINTED IN GREAT BRITAIN BY
THE ABERDEEN UNIVERSITY PRESS LTD., ABERDEEN

PREFACE

It is generally admitted that the standard courses of pure mathematics as taught in schools and colleges are not well adapted to form part of the professional training of engineers. But mathematics must form a large and an essential part of this training; and there has arisen a large body of text-books written to satisfy the special needs of engineering students. As a rule, however, these text-books will be found to consist of extracts, more or less modified, from the standard courses, interpolated with examples from engineering subjects proper. It is inevitable that the result of this scissors-and-paste method lacks the essential virtues of the original courses, logical development and ease of comprehension. Our experience is that the undoubted difficulty which the engineer has with mathematics is due principally to his general difficulty in reasoning in mathematical language, and that his difficulties are not restricted to the calculus.

In this book, the outcome of the combined efforts of an engineer and a mathematician, it has been our aim and ambition to develop a course *ab initio*, treating engineering calculus as a subject of engineering. This has led us to endeavour to give a whole, logical and consistent in itself, in which the point of view has always been that of the engineer, has always been concrete rather than abstract. We approach each new development of the subject

through an engineering problem, so that first the need of a proposition may be felt, and then the proposition provided. Mathematical terms are defined more in every-day language than is usual, and a large amount of consideration is naturally given to typical engineering applications. Particular stress has been laid upon steps of reasoning which experience has shown to present obstacles to students.

The student will find throughout the book a number of exercises by which he may test the extent to which he has followed the various sections of the work. He is strongly recommended to work carefully through these, as it is only by continual practice that he can expect to become fully acquainted with the subject.

The marks (C), (M) and (E) which will be found alongside many of the examples and exercises indicate that the problems are of particular interest to civil and constructional, mechanical and electrical engineers respectively.

For those students who wish to obtain a fair idea of the principles of the calculus as applied to engineering problems, and who have not time to work systematically right through the book, we would recommend the following shortened course: Chapters I and II, IV, V, VI, VII, X.

We wish to express our thanks to Prof. Horace Lamb and Mr. W. W. F. Pullen for permission to include the tables of exponential functions and hyperbolic logarithms respectively, and to Mr. A. E. Monkcom, A.R.C.S., for assistance and suggestions with the proofs.

EWART S. ANDREWS.

H. BRYON HEYWOOD.

LONDON, August, 1914.

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SUMMARY OF RESULTS ASSUMED KNOWN.

The following are a few of the more important results that should be known by a student reading this book. They are not intended, however, to represent a complete summary of a preliminary course in Mathematics:—

I. Algebra.

(1) *Representation of quantities by symbols.* (This, however, is developed at length in Chaps. II and III.) Calculation from formulæ.

(2) *The simple rules*—addition, subtraction, multiplication and division—and their applications. The student should be able to deal with easy fractions, factorisations, etc.

(3) *Equations*, linear and quadratic, and linear simultaneous equations. Algebraic and graphical solutions of equations.

(4) *Indices.*

$a^{\frac{r}{q}}$ means $\sqrt[q]{a^r}$, the q^{th} root of a^r .

The laws of indices.

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$a^0 = 1.$$

(5) *Logarithms.*

Laws. $\log(a \times b) = \log a + \log b$

$$\log(a \div b) = \log a - \log b$$

$$\log a^m = m \log a.$$

$$\log_a c = \log_a b \times \log_b c.$$

Numerical calculations with logarithms.

II. Geometry.

(1) *Triangles*, especially congruence (i.e. equality in all respects) and the following propositions:—

(a) The sum of the angles of a triangle is two right angles.

(b) *Pythagoras' Theorem*—if c is the hypotenuse and a and b are the sides of a right-angled triangle, then $c^2 = a^2 + b^2$.

(2) *Parallelograms*.

(3) *Circles*—chords, arcs, tangents, and the proposition,

If two chords AOB, COD intersect at O, then

$$AO \cdot OB = CO \cdot OD.$$

(4) *Ratio and Proportion*, simple results, especially—

A parallel to the base of a triangle divides the sides proportionally.

And the converse.

Similar figures are figures having the same shape: they can be placed in perspective.

(5) *Mensuration* :—

(a) Triangle: $\text{area} = \frac{1}{2}bh = \frac{1}{2}s(s-a)(s-b)(s-c)$
 $= \frac{1}{2}ab \sin C.$

(b) Trapezium: $\text{area} = \frac{1}{2}(a+b)h.$

(c) Circle: $\text{circumference} = 2\pi R$
 $\text{area} = \pi R^2.$

(d) Prism (or prismatic cylinder):

$\text{volume} = \text{height} \times \text{base}$

$= \text{slant height} \times \text{right section}.$

(e) Cone: $\text{volume} = \frac{1}{3} \text{base} \times \text{height}.$

(f) Sphere: $\text{area} = 4\pi R^2$
 $\text{volume} = \frac{4}{3}\pi R^3.$

(g) Spherical Cap: $\text{volume} = \frac{1}{3}\pi h^2(3R-h).$
 $(h = \text{height of cap}).$

III. Trigonometry.

(1) The measurement of angles in degrees and radians (circular measure).

(2) *Definitions* of the trigonometric ratios :—

$\sin \theta, \cos \theta, \tan \theta, \operatorname{cosec} \theta, \sec \theta, \cot \theta.$

(3) *Formulae*.

$$\cos^2 \theta + \sin^2 \theta = 1.$$

$$\tan^2 \theta = \sec^2 \theta - 1 \quad \cot^2 \theta = \operatorname{cosec}^2 \theta - 1.$$

$$\begin{aligned} \sin \left(\frac{\pi}{2} - \theta \right) &= \cos \left(\frac{\pi}{2} + \theta \right) = \sin (\pi - \theta) \\ &= \sin (2\pi + \theta) = -\sin (-\theta). \end{aligned}$$

$$\begin{aligned} \cos \theta &= \sin \left(\frac{\pi}{2} - \theta \right) = \sin \left(\frac{\pi}{2} + \theta \right) = -\cos (\pi - \theta), \\ &= \cos (2\pi + \theta) = \cos (-\theta). \end{aligned}$$

$$\tan \theta = \cot \left(\frac{\pi}{2} - \theta \right) = -\tan (\pi - \theta).$$

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1$$

$$= 1 - 2 \sin^2 \theta.$$

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

(4) The use of *Trigonometric Tables*. The following special ratios.—

| θ | 0° | 30° | 45° | 60° | 90° |
|---------------|-----------|----------------------|----------------------|----------------------|------------|
| $\sin \theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos \theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| $\tan \theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | Infinite |

(5) When θ (measured in radians) is small, $\sin \theta$ and θ approach each other in value, thus —

$$\sin \theta \approx \theta \quad (\text{approx. for } \theta \text{ small}),$$

and the fraction $\frac{\theta}{\sin \theta}$ approaches unity.

The value of the tangent also approaches the value of θ . These facts are illustrated by the following table —

| Degrees. | Radians (θ). | $\sin \theta$. | $\frac{\theta}{\sin \theta}$. | $\tan \theta$. |
|------------|-----------------------|-----------------|--------------------------------|-----------------|
| 20° | ·3491 | ·3420 | 1·0207 | ·3640 |
| 10° | ·1745 | ·1736 | 1·0049 | ·1763 |
| 5° | ·08727 | ·08716 | 1·0013 | ·08749 |
| 1° | ·0174533 | ·0174524 | 1·0001 | ·0174551 |

It will be noted that, when the angle is less than 90° ,

$$\sin \theta < \theta < \tan \theta.$$

(6) The following properties of a triangle— a, b, c are the sides of the triangle, and A, B, C are the opposite angles:—

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

THE CALCULUS.

CHAPTER I.

THE CALCULUS—INTRODUCTION AND GRAPHICAL TREATMENT.

§ 1. **Physical Quantities and Numbers.**—The physical quantities with which the engineer mostly has to deal are length, area, volume, time, mass, speed, acceleration, momentum, force, energy or work, power, stress (especially pressure, compression or tension, and shearing stress), and the electrical quantities,—quantity of electricity, resistance, voltage or potential, capacity, electric force, quantity of magnetism, magnetic force, inductance. All these are measurable and can be represented by numbers. It is very important to notice that all measurement is approximate, not exact.

We will take the simplest quantity, length, and what is said of that applies to all. If we use an ordinary scale we can read a length to the nearest $\frac{1}{16}$ in. or by estimation to the nearest $\frac{1}{100}$ in.; if we use a micrometer gauge, however, we can read, say, to the nearest $\frac{1}{1000}$ in., and physicists carry measurement to a much higher degree of accuracy. The reading obtained in the second case would be expressed for instance by 6.43 in., and in the third, 6.427 in.

1

In any reading the last figure expresses the limit of accuracy *either* of which our instrument is capable *or* which we require for our purpose. It is especially important not to omit final ciphers; thus 6.40 should never be written 6.4, for the latter would imply that we were incapable of reading to hundredths of an inch, so that the reading might really be, for instance, 6.37 or 6.42.

As we have already said, our limit of accuracy may not be the delicacy of our instruments, but the special requirements we have in view. For example, if we were constructing a connecting rod, say 6 ft. long, we should have to make our measurement to the nearest $\frac{1}{100}$ in., but if we were estimating the amount of tar necessary for a path about 6 ft. wide, the nearest inch would be more than accurate enough. This does not mean that we could not measure the path more accurately; doubtless we could measure it to a hundredth of an inch with proper precautions; but the accuracy is sufficient for our purpose.

ERRORS.—Very often we are not quite certain of the last figure of our reading. Such would be the case if we were measuring with a scale graduated to tenths of an inch, and were estimating to hundredths. A reading 6.43 might really be 6.41 or 6.44, say. The discrepancy is called the *error*. In some cases the error may be eliminated or reduced by taking the average of several readings, but it must be borne in mind that taking the average will never give another decimal place; *the average of ten scale readings estimated to hundredths will not give the result correct to a thousandth.*

The error mentioned above is the *absolute error*.

What is usually more important is the *proportional error*, that is the ratio of the absolute error to the reading. The error on the reading 6.41 of the correct value 6.43 is $-.02$, and the proportional error is $\frac{-.02}{6.43} = -.003$. [It would come to the same

thing if we took $\frac{-.02}{6.41}$.] Usually the percentage

error is taken, i.e. $100 \times$ proportional error. in this case $-.3$. We have a good example of this kind in the case of I beam and like rolled sections used in constructional steelwork. The makers cannot guarantee the sizes of these sections within $2\frac{1}{2}$ per cent. of the standard values. It would therefore be quite useless to calculate the safe load upon a girder as 12.3728 tons, because in that calculation we have assumed that the section is *exactly* as listed, but it is probably not quite, so that 12.4 tons would be all that it is worth while giving as the safe load.

In this connexion we should note that it is always *the significant figures that count, not the decimal point*.

If, for instance, we are measuring the extension in a steel bar 30 in. long, and it comes to .0176 in. and we want the *strain* (i.e. extension per unit length),

we say strain = $\frac{.0176}{30} = .000587$; this will then

have the same percentage error as our original measurement although there are more places of decimals.

§ 2. **Mathematical Formulæ.¹ The Variation and Interdependence of Physical Quantities.**—It

¹The student is supposed to know a little elementary algebra—see the summary of early work.

is convenient for the sake of brevity to use single letters for physical quantities, and thereby express the relations between them in a compact form. For instance, for a rectangle we have

$$\text{length} \times \text{breadth} = \text{area},$$

and this may be written

$$l \times b = A, \text{ or simply } lb = A,$$

if we agree that l shall stand for *length*, b for *breadth*, and A for *area*. Or again take the case of a rod of constant cross section stretched by hanging a given weight to its end; the elongation is proportional to the weight and the length of the rod, and is inversely proportional to the cross section. This can all be expressed by the formula

$$\delta = \frac{Wl}{EA} \dots \dots \dots (1)$$

where δ is the elongation, W is the weight applied, l is the length of the rod, A is its cross section, and E is a "numerical constant" (Young's Modulus).

These formulæ express laws connecting physical quantities. To work out a result for any practical case, we must "put into the formula" the numerical values for this case.

VARIABLE QUANTITIES.—So far we have supposed that we had only one isolated case to deal with, for example the elongation of a bar of given dimensions under a given weight. But we have also to think of instances where the physical quantities take a whole range of values. In the cylinder of a steam or gas engine, the volume and pressure vary continuously, and we are concerned with it not in merely one or two special positions but in every position. The

volume and the pressure are variable quantities or *variables*. • To take some other cases, the temperature of the air is a variable; it varies with ¹ the time, i.e. from hour to hour and from day to day; the output of an engine varies with the fuel consumed; the electric current in a circuit varies with the voltage. We could consider the example mentioned above (1) as a variable case; we might suppose the weight W to increase continuously, then the elongation δ would vary with the load.

In all these cases we might consider one variable as *independent*, the other as depending on it. The temperature depends on the time in this sense, that if the time is given then the temperature is determined; at 2 o'clock the temperature was 60° , at 2.30 it was 62° and so on (this does not imply a physical relation between temperature and time, but only a correspondence). And similarly with the other cases—the pressure in a cylinder is determined at each position of the piston, so that the pressure is dependent upon the volume. In these two cases respectively the time and volume will be spoken of as *independent variables*, the temperature and pressure as *dependent variables*. But this distinction is largely a matter of convenience: we might quite well, and sometimes do, consider the volume as dependent upon the pressure in the second example, for is not the volume determinate for each given pressure?

EXAMPLES ON VARIABLES.—In most engineering

¹ We must distinguish between the two expressions "varies as" and "varies with". The former implies the proportionality of the two quantities; the latter does not imply any particular manner of varying.

pocket-books will be found the formula for width of shaft keys

$$b = \frac{1}{4}d + \quad ,$$

where b is the breadth of the key in inches, and d is the diameter of the shaft in inches.

For any given value of d we can find b by means of the formula; b is therefore a *dependent variable*, because its value depends on the value of d that we are using. We shall have a number of examples of similar formulæ as we proceed in the book.

EXERCISES 1A

The following exercises are intended to give the student practice in the use of formulæ and the transformation of formulæ from one form into another.

1. The horse-power transmitted by a belt is given by the formula $\text{H.P.} = \frac{\pi n d (T_1 - T_2)}{33000}$, where n is the number of revolutions per minute, d the diameter of the pulley in feet, and T_1 and T_2 the tensions in lb. on the tight and slack sides respectively. If $T_1 = 2 T_2$ find the tensions when $n = 100$, $d = 4$, and H.P. = $3\frac{1}{2}$.

2. The resistance of electric circuits in parallel is given by the formula $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \dots$ where R = resistance of whole circuit and r_1, r_2 , etc., are the resistances of the separate branches. Find R when $r_1 = 10$ and $r_2 = 20$ ohms in the case of two branches.

3. Using the formula of the previous question, if $r_1 = 15$ ohms, what must r_2 be to make $R = 10$ ohms?

4. Given that 1 in. = 2.54 cm. and 1 kilo. = 2.205 lb., find how many kilos. per sq. cm. correspond to 1 lb. per sq. in.

5. A horse-power is measured by 33,000 foot-lb. per minute. If a gram-centimetre is 981 ergs and 10,000,000 ergs per second make a watt, find how many watts correspond to 1 horse-power. Use the data of question 4.

6. If 42,000,000 ergs are equivalent to the heat required to raise the temperature of 1 gram. of water 1° centigrade, how many foot-lb. will be required to raise 1 lb. of water through 1° Fahr. (a range of 100° C. is equivalent to one of 180° F.)? Use the data of questions 4 and 5.

7. The equation to a parabola is of the form $y = ax^2 + bx + c$. When $x = 0$, $y = 0$, when $x = \frac{L}{2}$, $y = r$, and when $x = L$, $y = 0$; express y in terms of x , L and r .

8. If $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \times 3} + \frac{x^4}{2 \times 3 \times 4} + \dots$ calculate its value to three decimal places when $x = .2$.

9. In the formula $y = ax^n$, $y = 2.34$ when $x = 2$, and $y = 20.62$ when $x = 5$. Find a and n .

10. The absolute temperatures of a gas before and after adiabatic expansion are given by the formula

$$\frac{T_1}{T_2} = \left(\frac{V_2}{V_1} \right)^{\gamma - 1}.$$

If $\frac{V_1}{V_2} = \frac{1}{6}$ and $\gamma = 1.408$, find T_2 if $T_1 = 531^{\circ}$ F. (abs.).

11. The mean pressure in a steam engine cylinder under isothermal expansion is given by the formula

$p_m = \frac{p_1}{r} (1 + 2.3 \log_{10} r)$, where p_1 is the initial pressure and r is the ratio of expansion. Find p_m if $p_1 = 120$ lb. per sq. in., and $r = 6$.

12. The flow of water in cubic feet per second over a triangular notch, the angle of which is a right angle is given by the formula $Q = \frac{8}{15} C_d \sqrt{2g} H^{5/2}$.

where C_d is the "coefficient of discharge" and H is the head in feet. Find the flow if $C_d = .60$ and the head is $15\frac{1}{2}$ in.

13. If $e = 2.718$, find e^e .

14. If $y = 3 \sin x + 4 \cos x$, find y for $x = 120^\circ$.

15. The ratio of the tensions T_1 and T_2 on the tight and slack sides of the belt running over a pulley is given by the formula: $2.3 \log_{10} \frac{T_1}{T_2} = \mu \theta$,

where μ is the coefficient of friction and θ is the angle of lap in radians of the belt upon the pulley. If $T_1 = 480$, $\mu = .4$, and $\theta = 120^\circ$, find T_2 .

16. The efficiency of a series of n pulleys is equal to η^n where η is the efficiency of each pulley. If η is .94, find the least number of pulleys that must be employed if the efficiency of the series is less than 50 per cent.

17. The horse-power absorbed in friction in a frusto-conical pivot is $\frac{2\pi NT}{33000}$ where

$$T = \frac{2\mu W (R^3 - R_1^3)}{12 \times 3 \sin \alpha (R^2 - R_1^2)}$$

Find the horse-power if $N = 140$, $R = 2$, $R_1 = .75$, $\mu = .08$, $\alpha = 30^\circ$ and $W = 2500$.

18. 50 cells the voltage of each of which is 1.8 volts are connected in series and the circuit is completed by a wire of 15 ohms resistance. If each cell has an internal resistance of .3 ohms, calculate the current in the circuit by the formula

$$C \text{ (amperes)} = \frac{E \text{ (volts)}}{R \text{ (ohms)}}$$

where E = total voltage and R = total resistance.

19. Using the result of question 5, find the horse-power absorbed by an arc lamp using 9 amperes and having a voltage drop of 50 volts by the formula

$$\text{Watts} = \text{amperes} \times \text{voltage drop.}$$

20. Hutton's formula for wind pressure on inclined surfaces is •

$$P_{\theta} = P_v \sin \theta^{1.84 \cos \theta - 1},$$

where P_v = pressure on a vertical surface, and θ is the angle of inclination of the surface to the horizontal. Find P_{θ} if $P_v = 50$, and $\theta = 30^\circ$.

§ 3. The Expression of Physical Relations by means of Graphs.—The representation of the dependence of two variable quantities by means of a curve will already be familiar to the student, for even

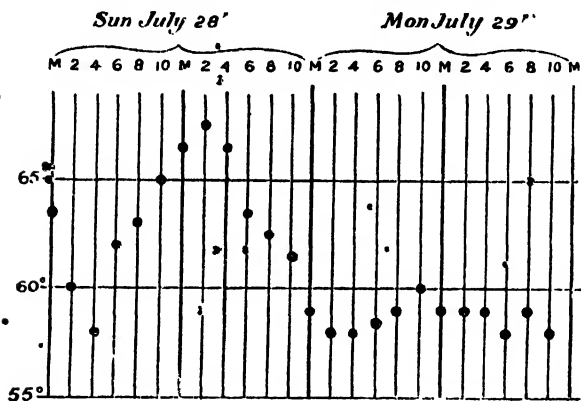


FIG. 1.—The above chart is constructed to show the thermometric records for two days, the dot indicating the position of the thermometer every two hours. From "The Daily Telegraph," 30 July, 1912.

if he has not done class work in "graphs," he will have seen examples daily—the weather charts in the papers, the diagrams showing growth of population, of exports and imports, and so forth. Let us take a weather chart as a first example.

In this chart observations have been taken at

intervals of two hours. We cannot get an exact temperature curve from this, as we do not know the temperature between each pair of observations. However if we draw a smooth curve through the dots,¹ we shall obtain a curve which differs very slightly from the true one. In making this statement we rely on the fact that the temperature does not vary in a sudden and irregular manner in intervals so short as two hours: from 6 a.m. to 10 a.m. on Sunday the temperature was rising; it was 63° at 8 a.m. and 65° at 10 a.m.; at 9 a.m. no doubt it was not very far from 64° .

This process is called *interpolating*: a good deal of judgment must be exercised in doing it. The

¹ The art of drawing a good smooth curve through a number of points is very valuable to an engineer and one rather difficult to acquire. It will assist to remember that the piece of the curve which lies between two consecutive points is affected by other points on either side. This is seen in the diagram. Be-

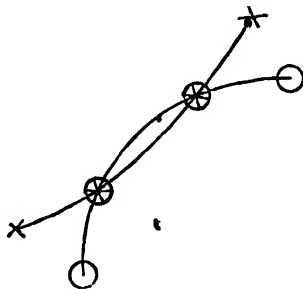


FIG. 2.

fore drawing in a piece of the curve freehand, the pen or pencil should be run two or three times just over the points in order to accustom the hand to the shape. A "French curve" may also be used.

nature of the data given, and the nature of the graph required must be taken into account. For instance in some cases the data are known to be not very exact while the graph is known to be smooth (as in a "stress-strain" curve for a metal bar, for instance); the graph need not then be drawn through the actual points, but should strike a smooth medium path amongst them. In other cases, although the data are exact, we do not require the exact graph with all

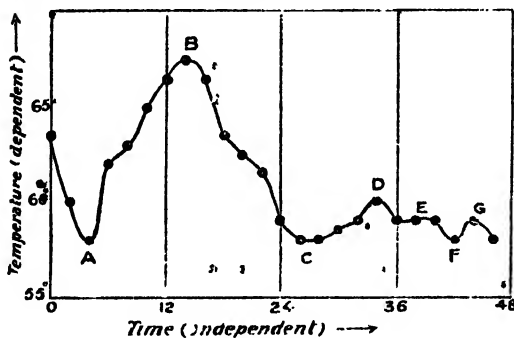


FIG. 3.

its irregularities but something to show us the general tendency of the dependent variable. Such modification of the data is dangerous, however; in the present instance we have relied implicitly on the "Daily Telegraph" and have drawn the curve strictly through the dots.

SLOPES AND RATES.—Notice that *when the curve slopes upwards the temperature is increasing, and the steeper the slope, the more rapid is the rate of rise.* When the curve slopes downwards the temperature is diminishing.

CRITICAL VALUES.—When the curve is horizontal.

the temperature is neither rising nor falling, i.e. it is *stationary*. Such points occur at ABCDEFG (Fig. 3); the temperature here is said to take *critical values*.

At B, D, G the temperature has *maximum values*: they are preceded by a *rise* and followed by a *fall*.

At A, C, E the temperature has *minimum values*: these are preceded by a *fall* and followed by a *rise*.

At F is a *point of inflexion*: it is preceded and followed by a fall. A point of inflexion clearly may also be preceded and followed by a rise. At a point of inflexion the curve changes from concave to convex or vice versa.

These ideas are of the utmost importance, being at the basis of the calculus: we shall use a more convenient curve for their numerical discussion.

§4.—Slopes.¹ MEASUREMENT OF SLOPES AND RATES; EXAMPLE OF STEAM-ENGINE MECHANISM.—In the following curve, the *abscissæ* (i.e. distances measured along the horizontal *axis* OX) represent the time, while the *ordinates* (i.e. distances measured vertically from OX) represent the displacements of a point (the cross-head in the steam-engine mechanism, see p. 93). The rate of increase of the ordinate measures the movement of the cross-head in a unit of time and so is the speed of the point.

Take any point M on the axis OX, and let MP be the corresponding ordinate. Let us represent the length OM by the letter t , and the length MP by y .

¹ *Slopes* or *Rates* are called also *Differential Coefficients* and *Derivatives*. These words all mean what amounts to the same thing but in different connexions. We shall later on use the term *Differential Coefficient* as it is the one in most general use.

Then t is the time which elapses from the starting-point, and y is the displacement. Take now another ordinate NQ : MN is the increase or *increment* in the time, and if we draw PK horizontally, KQ is the increment in the displacement. We shall use the symbol δ to denote "a change in" and then we may write

$$MN = \delta t; \quad KQ = \delta y.$$

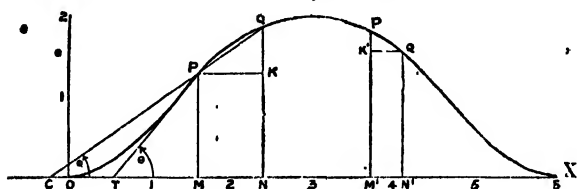


FIG. 4. Displacement curve for steam engine mechanism: connecting rod $3\frac{1}{2} \times$ crank. Ordinate = displacement of cross head: 1 unit = crank length. Abscissa = time or angle through which crank has turned: 1 unit = 60° .

Now as the average rate of increase of a quantity is equal to the total increase divided by the time, we have:—

$$\begin{aligned} \text{average rate of increase of } y &= \frac{\delta y}{\delta t} \\ &= \text{average speed of cross-head.} \end{aligned}$$

If the chord PQ meets the axis OX in C , and if we write $MCP = \phi =$ angle the chord PQ makes with the axis of x , then we have $KPQ = MCP = \phi$, since PK is parallel to CM .

Hence average rate of increase of y

¹ Note very carefully that δ is not a symbol denoting quantity: it is more of the nature of the signs $+$, \times , etc. δy is itself one single quantity, the increment in y .

$$= \frac{\delta y}{\delta t} = \frac{KQ}{PK} = \tan \phi.^1$$

A glance at the figure shows that the average rate of increase is less than the actual rate at the point *P* (where the curve is steeper). Suppose now that the ordinate *NQ* moves up towards *MP*. It is evident that the average rate approximates to the actual rate at *P*. And if *Q* were sufficiently near to *P*, the average rate and the true rate at *P* could be considered as identical. At the same time the chord *CPQ* would become indistinguishable from the tangent at *P* to the curve. Calling θ the inclination *MTP* of the tangent at *P* to the axis of *x*, we must therefore have

True rate of increase at *P* = $\tan \theta = \frac{\delta y}{\delta t}$ (when δt and with it δy become very small).

This we define as the *slope of the curve* at *P* and we use for it the notation

$$\frac{dy}{dt}$$

This slope is in essence a ratio but it is best for most purposes to regard $\frac{dy}{dt}$ as being *in one piece*, the parts *dy* and *dt* not having any separate meaning, while $\frac{\delta y}{\delta t}$ on the other hand is a fraction pure and simple.

We are thus able to find a rate from a graph—in this case, a speed from a displacement curve. We

¹ See summary of early work for the meaning of $\tan \phi$.

have only to draw the tangent to the curve at the point in question and find its slope.¹

NEGATIVE RATES.—The right hand half of the curve descends, or slopes downward. If we take an ordinate $M'P'$ and a consecutive ordinate $N'Q'$ in the same way as before, the second ordinate is shorter than the first. *The increment or change in the ordinate is a negative quantity, and is equal to minus $K'P'$, i.e.*

$$\delta y = - K'P'.$$

The rate or slope is obtained as before, and it is still $\tan \theta'$ or $\frac{dy}{dx}$, but it has a negative value; the angle θ' is the angle which the tangent at P' makes with the axis of x and is, according to the usual convention, considered negative.

ZERO RATES.—At the summit of the curve and also at O and x , the slope, or rate of increase is zero,

¹It is not easy to draw a tangent to a curve at a point *accurately*. A good plan is to mark off two near points K, L (Fig. 5), on the curve equidistant from P . Then KL gives a better direction than an attempted tangent. The slope may be found conveniently by finding the rise between two points on the line on ordinates say 10 units apart. The slope need not always be measured numerically, but the length representing the slope can be picked off with dividers,

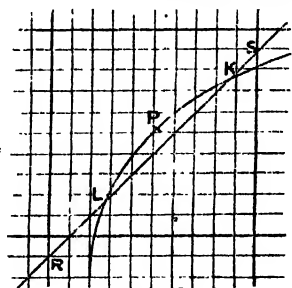


FIG. 5.

$$\text{i.e. } \frac{dy}{dx} = 0.$$

This clearly holds at all critical points.

§ 5. **Derived Curves.**—We can find the rate at any point of the curve in the above section: the rate then is a variable quantity which takes up one value at each value of the abscissa, and therefore it can be graphed in a similar diagram. Such a curve is called a derived curve (Fig. 6). In this case it is a *curve of speeds*—a velocity curve.

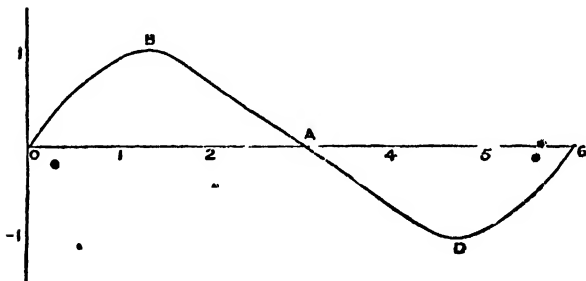


FIG. 6.

It will be observed in Fig. 6 that the slope is initially zero, that it is most rapid in the neighbourhood of B, then diminishes down to zero at A. It then becomes negative, the greatest negative value occurring at D and finally the slope vanishes again at 6. Note that the derived curve crosses the axis at the critical points of the original curve.

The derived curve is obtained graphically by finding the slope at a series of points on the original curve, and so obtaining a series of points on the new curve. It is one of the most unsatisfactory processes in graphics, but (when the law is not given by a

formula so that the methods of Chapter III do not apply) there is no other method of attaining the result.

.. SECOND DERIVED CURVES.—Starting with the de-

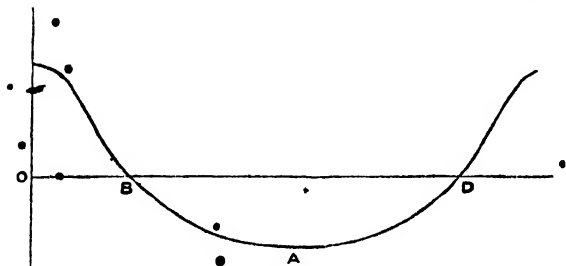


FIG. 7.

rived curve we can find its slope at each point, and so obtain its derived curve. The second derived curve will express the rate of change of the rate of change; in the case in point it will express the rate

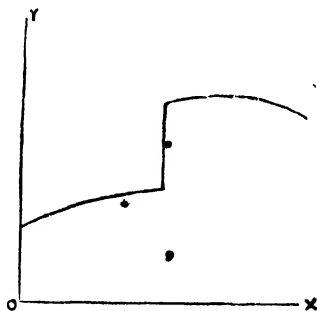


FIG. 8.

of change of the speed and will therefore be the acceleration curve (Fig. 7).

Note.—Curves are sometimes met with which make a sudden jump, as in the figure. Such curves,

are called *discontinuous*: they have of course no "slope" at the points of discontinuity.

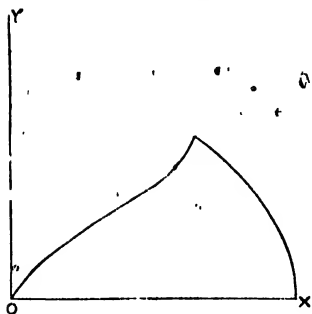


FIG. 9.

More frequently we find curves with sharp angles. In these the slope makes a sudden change at the angles, and is discontinuous, but the curve itself is continuous.

If a curve is discontinuous or if its derived curve is discontinuous we cannot express the quantities which it represents by a general formula.

EXAMPLES OF DISCONTINUITY.

Shear and Bending Moment Curves.—A very familiar example of a discontinuous curve in engineering work occurs in the case of shear diagrams for beams with isolated load systems. Fig. 10 shows such a case.

Now it can be shown that the shear diagram for a beam is the derived or slope curve for the bending moment diagram and that the load diagram is the derived curve for the shear curve. It is clear from the figure that the bending moment diagram has sharp angles so that we see that when a curve is discontinuous, it is derived from a curve with sharp angles. Further, the derived curve for a curve with sharp angles will consist of a number of isolated pieces; this is clear from the load curve which consists of a number of isolated forces or vertical lines.

Indicator Diagrams.---In theoretical indicator diagrams for showing the variation of pressure at dif-

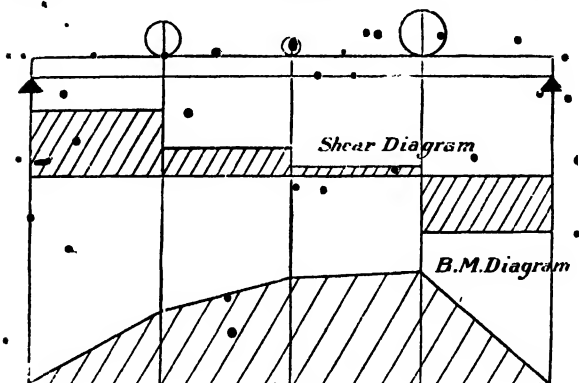


FIG. 10.

ferent points in the stroke of a steam engine (Fig. 11), we have another familiar example of a curve with

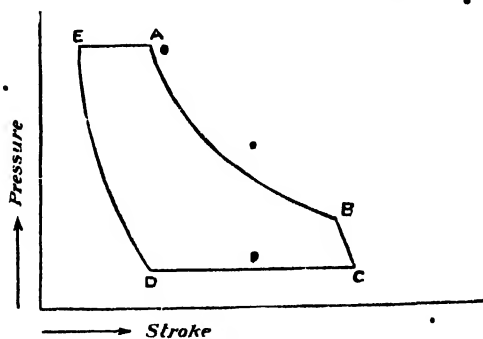


FIG. 11.

sharp angles; these points occur at ABCDE which represent the "cut off," "release," "exhaust," "compression" and "full admission" respectively.

§ 6. **Sum Curves.**—The following curve is a *crank-effort or force diagram*. We use it to illustrate curves of areas, *sum or integral curves*.

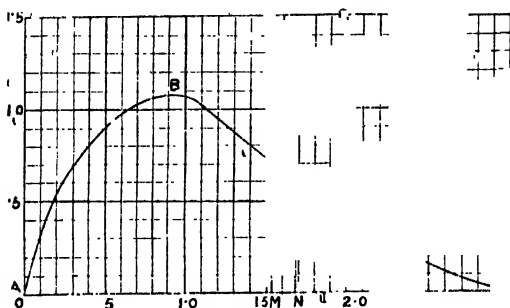


FIG. 12.—Crank-effort Diagram.

TABLE OF AREAS.

| $x =$ | 0 | .2 | .4 | .6 | .8 | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 | 2.2 | 2.4 | 2.6 | 2.8 | 3.0 |
|---------|---|-----|-----|-----|-----|-----|-----|------|------|------|------|------|------|------|------|------|
| areas = | 0 | .05 | .19 | .37 | .57 | .79 | .99 | 1.17 | 1.31 | 1.42 | 1.53 | 1.60 | 1.65 | 1.68 | 1.70 | 1.70 |

In this diagram the abscissæ represent the distance through which the crank pin moves up to a half revolution; the ordinate represents the crank-effort, a force acting upon the crank. Take any ordinate MP , and an ordinate NQ very near to it. When the crank pin moves through the distance MN , the piston will do work equal to

$$\text{average effort} \times MN.$$

Now MP and NQ are very nearly equal, "so that the work done is approximately"

$$NQ \times MN = \text{rectangle } NK.$$

The rectangle is practically a slice of the curve, if we neglect the triangle at the top of the rectangle. So that the *work done is equal to the area of the curve between the two corresponding ordinates.*

This applies to the motion through a very short distance MN; if we want the work done by the piston while the crank pin moves from A to M, we shall only have to add together all the similar slices of the curve for a number of short motions making up AM. Thus the work done by the piston while the crank pin moves from A to M is equal to the area under the curve from A to M. This is an *exact* result, for we can make the short motions like MN as small as we please, and hence the row of triangles like QPK will form a vanishing area lying along the curve: in fact these triangles occur only in the proof and not in the result.

We can now draw a second curve the *sum*¹ curve (Fig. 13), in which the abscissa are the same as before, and the ordinate is *the area of the original or primitive curve up to the corresponding point.*

The student will notice that the sum curve must start at zero at A and rise continuously; also that it will move rapidly when the ordinate of the original curve is longer, so that *area* is being added on faster, in fact its rate of rise or *its slope is proportional to the ordinate of the original curve.* This fact is very important and we shall return to it.

We have described this process for a special curve—crank-effort or force diagram—but it may evidently be applied to any curve, whatever its significance

¹ The reason for calling this a sum curve is apparent already, and will be explained later.

As a second example we will take a velocity curve: In the following diagram the abscissa represents the

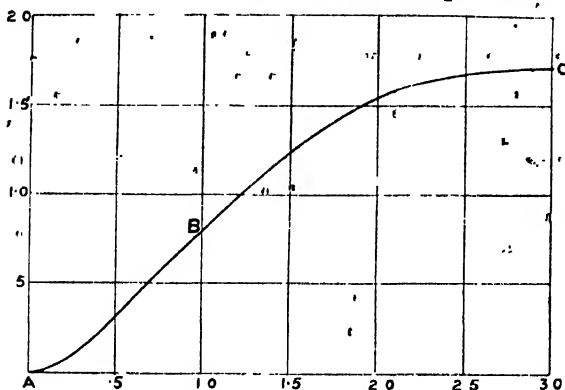


FIG. 13. Work Curve Diagram showing Work Done,

time, and the ordinate is the velocity of a point at that time.

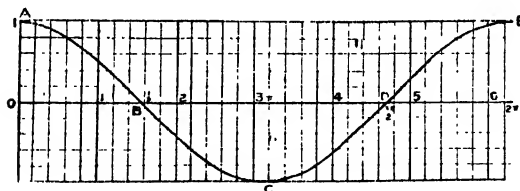


FIG. 14.—Velocity Curve. [This curve is the cosine curve, and represents the velocity in harmonic motion.]

Now the distance covered by a moving point is the average velocity multiplied by the time: for a very short period of time the variation of the velocity may be neglected and the short distance described is this velocity multiplied by the short interval of time. The student will see that if we measure from the

starting-point, the displacement of the moving point is given by the sum curve, which is therefore the displacement curve.

It will be observed that so long as the velocity curve is positive, the displacement curve rises. At B the ordinate of the velocity curve becomes negative, and with it the added area. (Areas below the axis are negative.) The moving point comes to rest at B and begins to move backwards. So the sum curve ceases to rise and begins to fall; it continues to fall so long as the velocity is negative. It will be remarked that after C, the total area of the velocity

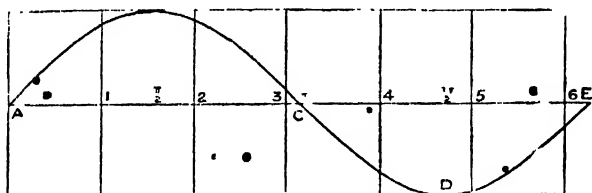


FIG. 15.—Displacement Curve.

curve is negative, since the negative area below the axis exceeds the positive area above it, and consequently the sum curve is negative.

§7.—Connection between sum curves and derived curves. We shall now prove, what the student has already suspected, that if one curve is the derived curve of another, then the second is the sum curve of the first. We shall use a pair of curves that we have already discussed, namely Figs. 4 and 6 which we reproduce (Figs. 16 and 17).

The ordinate LR in Fig. 17 is approximately equal to $\frac{y}{\delta x}$ in Fig. 16 (in the system of units

adopted). Calling $LR = z$, we have then

$$z = \frac{\delta y}{x} \text{ (approximately).}$$

And we have to prove that the increment in

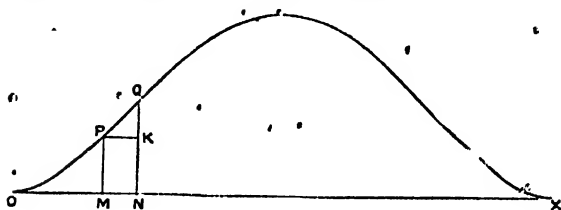


FIG. 16.

the ordinate in Fig. 16 is equal to an increment of area in Fig. 17. That is to say we must show that

$$\delta y = z \cdot x \text{ (approximately),}$$

and this is clearly true.

This result is, of course, not rigidly proved, since each of the equations is only an approximation; how-

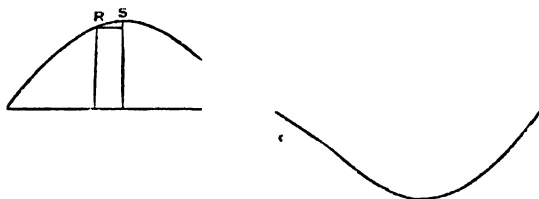


FIG. 17.

ever we have seen already that the shorter the interval considered is, the closer do these approximations become, and, by taking the intervals short enough, we can make the approximations as close as we please. Moreover we have also seen that all practical work is approximate in nature and not exact (§ 1, p. 1). Hence the mathematical result covers

all that we do, or ever could, require. The mathematical result is *exact*, but to prove this rigidly, it would be necessary to introduce mental processes¹ with which the engineering student is not, as a rule, familiar.²

§ 8.—**Measurement of Areas.**—**FIRST METHOD.**—**COUNTING SQUARES.**—To obtain the sum curve, the simplest method is to draw the original curve on squared paper, and to count the squares. Fig. 13 has been obtained from Fig. 12 in this way.

The determination of areas is very important, and we proceed to give several other methods: these methods are more accurate than the one already given, but (excepting the second) they are not so well adapted for finding a growing area at each stage. We shall meet with the method of *calculating* areas in Chapter V when we are doing symbolic integration.

SECOND METHOD.—The sum curve can be obtained graphically as follows. Let ACD, Fig. 18, be any primitive curve on a straight base AB. Divide AB into any number of parts, not necessarily equal (but for convenience of working they are generally taken as equal). These so-called base elements should be taken so small that the portion of the curve above them may be taken as a straight line. About 1 cm. or $\frac{1}{4}$ in. will usually be a suitable size and in most cases a smaller element will come at the end. Find the mid-points, 1, 2, 3, etc., of each of the base elements and let the verticals through these mid-

¹ Especially the method of Limits: this we shall, as a matter of fact, employ later on, but not in a rigorous form.

² Obtaining a sum curve is called *integrating*, the inverse process of obtaining a derived curve is *differentiating*: see later (p. 79).

points meet the curve in $1a, 2a, 3a$, etc. Now project the last points on a vertical line AE , thus obtaining the points $1b, 2b, 3b$, etc., and join such points to a pole P on AB produced and at some convenient distance p from A . Across space 1 then draw Ad parallel to $P 1b$; de across space 2 parallel to $P 2b$, and so on, until the point n is reached. Then the curve $A de \dots n$ is the sum curve of the given

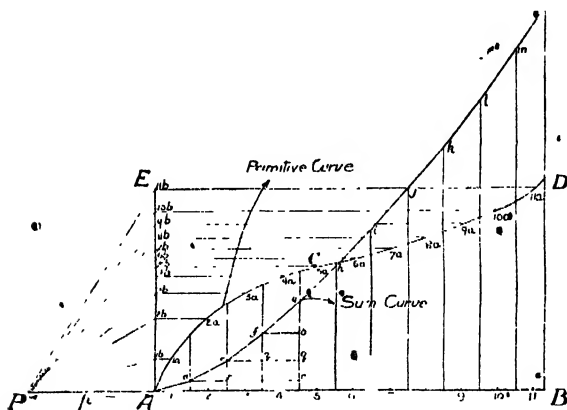


FIG. 18. Sum-curve Construction.

curve, and to some scale Bn represents the area of the whole curve.

PROOF.—Consider one of the elements, say 4, and draw fo horizontally

Now $\Delta f g o$ is similar to the $\Delta P 4b A$

$$\therefore \frac{go}{fo} = \frac{4b A}{PA}$$

but $PA = p$ and $4b A = 4 \cdot 4a$

$$\therefore go = \frac{fo \times 4 \cdot 4a}{p} = \frac{\text{area of element 4 of curve}}{p}$$

Similarly $f q = \frac{\text{area of element 3 of curve}}{p}$ and so on.

\therefore Ordinate through $g = g q + f q + \dots$

$= \frac{\text{area of first four elements of curve}}{p}$

\therefore The curve $A d e \dots h$ is the sum curve required.

Then if $B n$ be measured on the vertical scale and p be measured on the horizontal scale, the area of the whole curve will be equal to $p \times B n$.

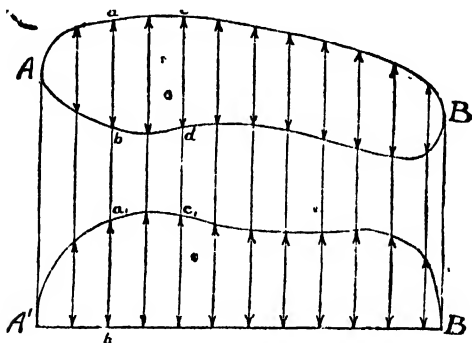


FIG. 19.

It is obviously advisable to make p some convenient round number of units.

The sum curve obtained by this method may have the same operation performed on it, and thus the second sum curve of the primitive curve is obtained, and so on.

If the operation be performed on a rectangle, the sum curve will obviously come a sloping straight line, and if the sum curve of a sloping straight line be drawn, it will be found to be a parabola (§ 12). In the case in which it is required to apply this construc-

tion to a curve which is not on a straight base, the curve is first brought to a straight base as follows:—

Suppose $AcBd$, Fig. 19, is a closed curve. Draw verticals through AB to meet a horizontal base AB . Divide the curve into a number of segments by vertical lines at short distances apart, and set up from the base AB lengths a_1b_1 , etc., equal to the vertical portions ab , etc., on the curve. Joining up the points thus obtained we get the corresponding curve $A'a_1c_1B'$, on a straight base.

THIRD METHOD. PLANIMETER.—This instrument is to be found in most drawing offices, and generally provides the quickest and most accurate method of measuring areas. The pattern in most general use is Amsler's.

A square rod A , Fig. 20, carries at its end a pointer E and has adjustably fixed to it a slide B to which is pivoted a second rod C . This rod carries at its end a weighted pin D which is pressed into the paper and thus fixed relatively thereto. The pointer E is first placed at some point X on the boundary of the area and the reading upon the graduated wheel F which runs over the paper is taken by aid of a vernier. The pointer is then moved carefully round the curve until the point X is again reached. The reading of the wheel is then taken again and the difference between the two readings gives the area of the curve. The measuring wheel F is geared up to a second wheel G . The scale to which the areas are read is adjusted by moving the slide B along the rod A . Students should endeavour to obtain practice in the use of this instrument.

The proof of the working of the instrument involves some rather advanced work which is beyond

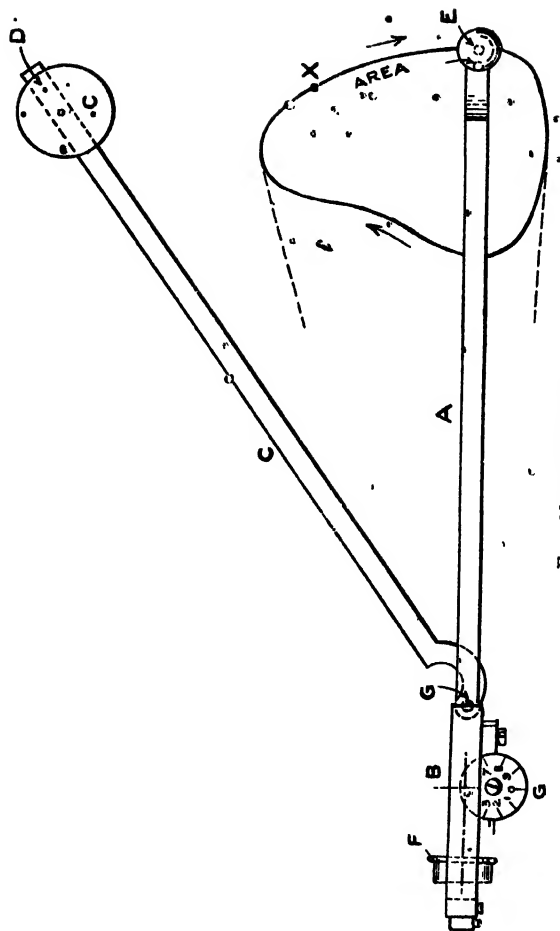


FIG. 20.—Planimeter.

our present scope and it is rather better to accept its accuracy without proof than to attempt a proof that is not really sound.

- **FOURTH METHOD: MID-ORDINATE RULE.**—The area of a curve is the average height (mean ordinate) \times the base.

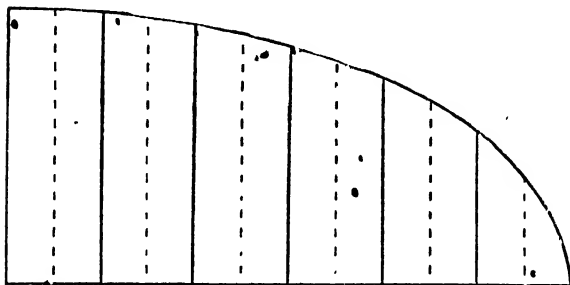


FIG. 21.

Subdivide the curve into vertical strips of equal breadth, and draw the *mid-ordinate* of each strip (the dotted lines). The average height of the curve is approximately the average of these dotted lines. The greater the number of strips the closer will the approximation be. If the length of the base is a , and there are n strips with mid-ordinates $= y_1, y_2, \dots, y_n$, then we shall have

$$\text{Area} = a \left(\frac{y_1 + y_2 + \dots + y_n}{n} \right)$$

$$= \frac{a}{n} (y_1 + \dots + y_n).$$

- The curve in the figure is a quarter of an ellipse. We have chosen intentionally a curve which becomes steep, and which therefore is not well adapted to Simpson's and some of the other rules—(see the tests applied to this curve below).

. FIFTH METHOD. SIMPSON'S RULE.—The area must be subdivided into an *even* number of strips (in this case 6). If the *seven* ordinates which bound these *six* strips are called $y_1, y_2, y_3, y_4, y_5, y_6, y_7$, and the *breadth* of each strip is δ , then

$$\text{Area} = \frac{\delta}{3} [y_1 + y_7 + 4(y_2 + y_4 + y_6) + 2(y_3 + y_5)]$$

δ
3 sum of extreme ordinates, plus four times

sum of even ordinates, plus twice sum of odd ordinates].

This is Simpson's *First* Rule: we do not think it worth while increasing the number of methods unduly by giving Simpson's Second Rule.

SIXTH METHOD. WEDDLE'S RULE.—This is the most accurate of this type of methods. The only objection to it is that the area must be subdivided into six equal parts, or a multiple of six and not any other number. Using the same notation as before the rule is

$$\text{Area} = \frac{3\delta}{10} [y_1 + y_3 + y_5 + y_7 + 5(y_2 + y_6) + 6y_4].$$

The following are the determinations of the area of the above curve by the several methods:—

| | | Error. | Percentage Error. |
|---------------------|--------|--------|-------------------|
| Exact value | 14.127 | nil. | — |
| 1. Counting squares | 14.09 | - 0.05 | - 0.3 |
| 2. | 14.2 | + 0.1 | + 0.7 |
| 3. Planimeter | 14.13 | 0.01 | 0.1 |
| (three readings) | 14.11 | - 0.03 | - 0.2 |
| | 14.16 | + 0.02 | + 0.2 |
| Mean | 14.13 | 0.01 | 0.1 |
| 4. Mid ordinates | 14.26 | - 0.12 | - 0.9 |
| 5. Simpson's Rule | 14.00 | - 0.14 | - 1.0 |
| 6. Weddle's Rule | 14.03 | - 0.11 | - 0.8 |

Note that on account of the "steepness" results for rules 4, 5, 6 are not very good: usually Weddle's rule shows a more marked superiority over the others than it does here.

Remarks on the last three rules.—Greater accuracy is always obtained by increasing the number of strips into which the curve is subdivided. These rules give the best results for curves which are smooth and which do not become steep.

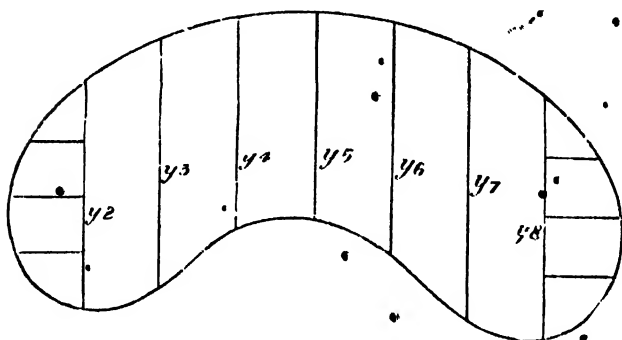
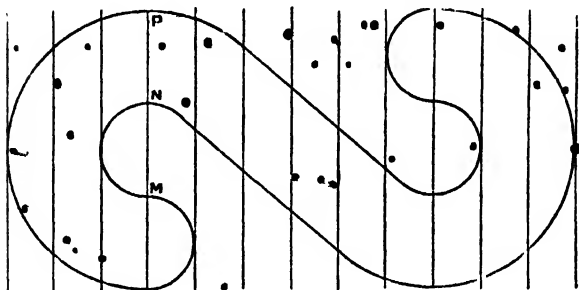


FIG. 22.

They apply equally well to *closed* curves (Fig. 22) which must be divided into strips of equal breadth as before. In the figure there are eight strips, and the end ordinates y_1 and y_9 both vanish. But at the ends the curve is *steep*, so it is better to separate the end strips and cut them up in the perpendicular direction.

The rules (especially Weddle's) will even apply to irregular areas in which the ordinates meet the bounding curve more than twice. In Fig. 23 the fourth ordinate LMNP meets the curve four times.

The corresponding length y_3 is the sum of LM and NP . Great accuracy must not be expected as the



• FIG. 23.

curve becomes steep several times, but the value obtained will be good enough for most purposes.

EXERCISES 1B.

1. x and t are the distances in miles and the time in hours of a train from a railway terminus. The slope of this curve at any point is a measure of the speed. What is the greatest speed and at what point does it occur?

| | | | | | | | | | | | |
|-----|---|-----|-----|------|------|------|------|------|------|------|------|
| x | 0 | 1.5 | 6.0 | 14.0 | 19.0 | 21.0 | 21.5 | 21.8 | 23.0 | 24.7 | 26.8 |
| t | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |

2. The following values of p and θ are given.

Find the slope $\frac{dp}{d\theta}$ when $\theta = 115$.

| c | p |
|-----|-------|
| 100 | 14.70 |
| 105 | 17.53 |
| 110 | 20.80 |
| 115 | 24.54 |
| 120 | 28.83 |
| 125 | 33.71 |
| 130 | 39.25 |

3. The following formula gives the increase c in volume of one pound of water in changing from water to steam at an absolute temperature t centigrade and pressure p lb. per sq. ft., L being the latent heat and J the mechanical equivalent of heat.

$$c = \frac{JL}{t} \cdot \frac{dt}{dp}$$

$\frac{dt}{dp}$ is thus the slope of the curve of t plotted against p . Find c at $t = 428$ given $L = 497.2$ and $J = 1293$, the relation between t and p being as follows:—

| t | p |
|-----|-------|
| 413 | 7563 |
| 418 | 8698 |
| 423 | 9966 |
| 428 | 11380 |
| 433 | 12940 |
| 438 | 14680 |
| 443 | 16580 |

4. The following numbers give the speed v of a train, in miles per hour and the time t in minutes from starting. Draw the sum curve which gives the distances passed through and find the total distance.

| | | | | | | | | | | | | | | |
|-----|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| v | 0 | 2.4 | 4.7 | 7.2 | 9.6 | 12.0 | 14.3 | 16.9 | 18.9 | 20.7 | 22.2 | 23.4 | 24.3 | 24.9 |
| t | 0.00 | 0.04 | 0.08 | 0.12 | 0.16 | 0.20 | 0.24 | 0.28 | 0.32 | 0.36 | 0.40 | 0.44 | 0.48 | 0.52 |

5. The area A square feet of the horizontal section of a reservoir at a height h from the lowest point is as follows:—

| | | | | | | | | | | | | | |
|-----|---|------|------|------|------|------|------|------|------|------|------|------|------|
| h | 0 | 2.5 | 4 | 7 | 10 | 12.5 | 15 | 17.5 | 20 | 23 | 25 | 28 | 30 |
| A | 0 | 2510 | 3400 | 4520 | 5160 | 5490 | 5810 | 6210 | 6390 | 7810 | 8270 | 8679 | 8780 |

Plot A against h and draw the sum curve, the ordinates of which will be a measure of the volume of water contained up to each height. What is the total volume and the change in volume if the height drops from 15.5 to 14.5?

6. The following figures give the wind pressure p in lb. per sq. ft. on a vertical plane surface at heights h in feet above the ground.

| | | | | | |
|-----|----|----|----|----|----|
| h | 5 | 15 | 25 | 35 | 50 |
| p | 13 | 22 | 24 | 27 | 31 |

What is the total wind force on a wall 100 ft. long and 40 ft. high?

7. An area is divided into ten equal parts by 11 equidistant parallel lines .2 in. apart, the first and last touching the bounding curve. The lengths in inches of these ordinates are: 0, 1.24, 2.37, 4.10, 5.28, 4.76, 4.60, 4.36, 2.45, 1.62, 0. Find by Simpson's rule the area in square inches.

8. A circle is drawn of 8 in. diameter. The diameter is divided into 8 equal parts and ordinates are

drawn at right angles to the diameter. Find the area by Simpson's rule, checking the ordinates by calculation for preference. What is the percentage error?

9. x being the distance in feet across a river 80 ft. wide, measuring from one side and y the depth of water in feet, the following measurements are made:—

| | | | | | | | | |
|-----|---|----|----|----|----|----|----|----|
| x | 0 | 10 | 25 | 33 | 40 | 48 | 60 | 70 |
| y | 0 | 4 | 7 | 8 | 10 | 9 | 6 | 4 |

Find the area of the cross section. If the average speed of the water normal to the section is 3.2 ft. per sec. find the flow in cub. ft. per sec.

10. In a determination of the area of a field, x is distance measured along a straight line AB from the point A; the values of y are offsets or distances, both in chains measured at right angles to AB to the border of the field. Find the average breadth of the field from the line AB to the border.

| | | | | | | |
|-----|------|------|------|------|------|------|
| x | 0 | 1.50 | 3.00 | 5.00 | 7.50 | 9.00 |
| y | 0.53 | 0.47 | 0.40 | 0.42 | 0.46 | 0.52 |

ANSWERS TO EXERCISES.

1A.

1. 183.2 and 91.6 lb. 2. 6.67 ohms. 3. 30 ohms. 4. .07 kilo. per sq. cm. 5. 746 watts. 6. 778 foot-lb. 7. $y = \frac{1}{L^2} x^2 (L - x)$. 8. 1.221. 9. $a = 0.4514$, $n = 2.374$. 10. 255° F. absolute.

11. 55.8 lb. per sq. in. 12. 4.85 cub. ft. per second.
 13. 1.284. 14. .598. 15. 207.7 lb. 16. 12.
 17. 1.31. 18. 3 amperes. 19. .603 H.P.
 20. 33 lb. per sq. in.

1B.

1. 80 miles an hour at 6 miles. 2. .80. 3. 5.45.
 4. 665 ft. 5. 174800 cub. ft.; 5810 cub. ft.
 6. 85200 lb. 7. 6.25 sq. in. 8. 49.33 sq. in.; 1.89
 per cent. 9. 467 sq. ft.; 1494 cub. ft. per second.
 10. .45 chains.

CHAPTER II.

LAWS EXPRESSED SYMBOLICALLY.¹

§ 9. **Formulæ.** To know the law connecting two physical quantities, x and y , say, where y is supposed dependent, we must know the value of y corresponding to every value of x . The law can always be expressed by a graph. But very often, especially when it is obtained theoretically, it can be put into a more definite form. For example, when two rough bodies are in contact, we know that the limiting friction between them is *proportional to the pressure*, a law which we express

$$F = \mu P.$$

Or again, for a mass of perfect gas at constant temperature, we know that the volume is *inversely proportional to the pressure* (Boyle's or Mariotte's Law), and this we express

$$p = \frac{k}{v}, \text{ or } pv = k.$$

Whenever a law, experimental or theoretical, can be put into a definite shape in this way, it can always be condensed into a *formula*. In fact most laws are too complicated to be expressed in words at all, and

¹ This and the next chapter may be omitted by students who have already a knowledge of graphs and the laws which they represent.

then a formula is our only resource besides a graph. Even when the law cannot be put exactly into this definite form, and as a matter of fact no experimental law can be exact, we can still find a formula which expresses a law approximating very closely to it. Such a law devised to fit a series of experimental data is called *empirical*. There is a great advantage in finding an empirical law, for not only does it give us in a very brief and convenient form what could otherwise only be expressed by a complicated table of values, or by a graph, but it often throws a great deal of light on the nature and relations of the quantities involved, and leads to new series of laws and results.

The student will notice that two important processes are involved. The first is to draw a graph from a formula; the second is, given a graph, to deduce an empirical formula.

We shall now give a list of the simplest formulæ which occur, including nearly all those that an engineer is likely to encounter, and discuss them briefly.

I. ALGEBRAICAL FORMULÆ.

§ 10. **Algebraical Laws.**—These are the laws which involve only the processes of addition, subtraction, multiplication, division, and extraction of roots.

We remind the student that there are two square roots of any number x , which are written $\pm \sqrt{x}$: either of them may be written \sqrt{x} , but either of the symbols $x^{\frac{1}{2}}$, \sqrt{x} written alone means the *positive*

square root. The symbol x^4 means the cube of the fourth root of x .

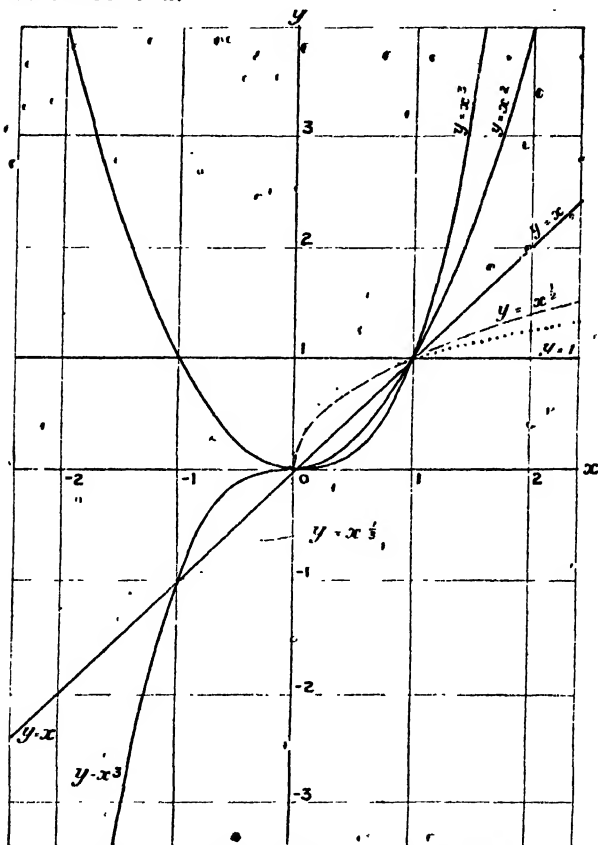


FIG. 24.—Graphs of $y=1, x, x^2, x^3, x^{1/2}, x^{1/3}$.

We shall now discuss a series of typical algebraic laws.

POWERS. POSITIVE INTEGRAL.—In Fig. 24 are

shown the graphs of some *positive integral* powers. The first graph $y = 1$ can be regarded as the *zero* power of x , for $x^0 = 1$. The second $y = x$ gives the first power, for which the ordinate is always equal to the abscissa. All the other graphs of positive integral powers touch the x -axis at the origin. The higher the power, the slower the curve rises at first, and the quicker it rises afterwards. They all pass through the point $x = 1, y = 1$. When x is negative, all the even powers, $y = x^0, x^2, x^4, \dots$ have positive values, and the odd powers $y = x, x^3, x^5, \dots$ have negative values. The even power graphs are all *symmetrical* about the y -axis, that is to say, the left hand side is obtained by reflecting the right hand side in the y -axis. The negative power graphs are *skew-symmetrical* about the origin, the left hand side being obtained by two reflexions of the right hand side, the first in the y -axis, and the second in the x -axis. This is expressed algebraically,

$$y = (-x)^2 = +x^2 \text{ for an even power}$$

$$\text{and } y = (-x)^3 = -x^3 \text{ for an odd power.}$$

POSITIVE FRACTIONAL POWERS.—It should first be noticed that for fractional powers in which the denominator is even, the curve has no left hand side: for example if in $y = \sqrt{x} = x^{\frac{1}{2}}$, we put a negative number for x , there is *no value* of y for there is no square root of a negative number.¹ The graph of $y = x^{\frac{1}{2}}$ is seen to stop short at the origin. When the denominator is odd there are values of y for

¹ In higher work we get roots of negative numbers, by *inventing* new numbers, called the *imaginary numbers*. The first of these is written i and signifies $\sqrt{-1}$. An imaginary number cannot be shown in an ordinary graph.

negative values of x . This is seen in the graph of $y = x^{\frac{1}{2}}$.

The graphs of the powers less than unity are above the straight line $y = x$, between $x = 0$ and $x = 1$, and beyond $x = 1$ they are below $y = x$. The graph of $y = x^{\frac{2}{3}}$ is intermediate between $y = x$ and $y = x^2$; the graph of $y = x^{\frac{3}{4}}$ is intermediate between $y = x^{\frac{2}{3}}$ and $y = x^{\frac{3}{2}}$.

The graph of $y = x^{\frac{1}{2}}$ may be written $y^2 = x$, and hence it is the same curve as the graph of $y = x^{\frac{3}{2}}$, from which it is obtained by interchanging x and y .

NEGATIVE POWERS. --For any negative power say $y = \frac{1}{x}$, the ordinate becomes indefinitely great, when the abscissa becomes indefinitely small (positive or negative). The curve does not cut the y -axis at all but gets indefinitely near to it at a great distance from O .

A straight line that a curve approaches indefinitely without ever meeting is called an *asymptote*.

The curves $y = \frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, etc., approach the y -axis in opposite directions, while $y = \frac{1}{x^2}$, $\frac{1}{x^3}$, etc., approach it in the same direction.

The curve $y = x^{-\frac{1}{2}}$ lies on one side of the asymptote, and approaches it in one direction only, as $x^{-\frac{1}{2}}$ does not exist for negative values of x .

The x -axis is also asymptotic to these curves.

The curve $y = \frac{1}{x}$ (or $xy = 1$) is especially important: it is a *rectangular hyperbola*. An important property of this curve is that the rectangle

contained by the co-ordinates (xy) is constant in area. The origin is the centre.

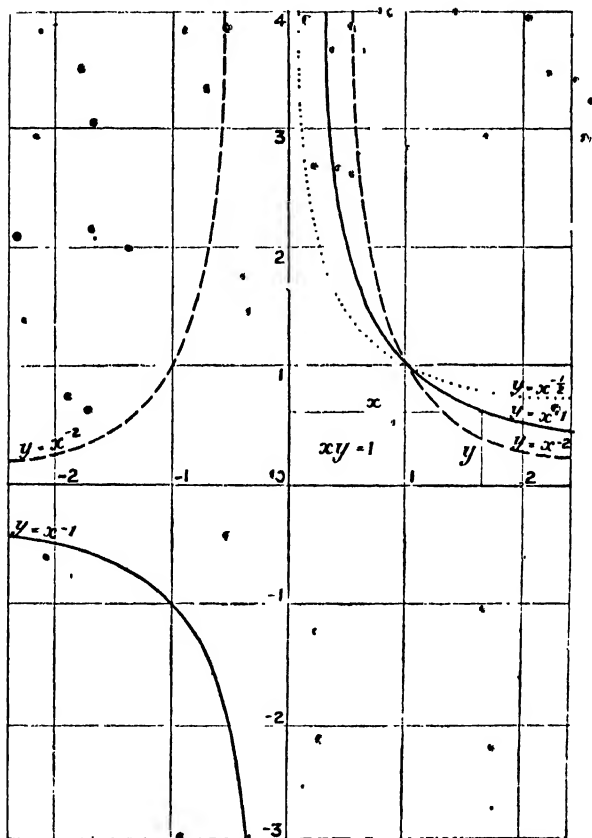


FIG. 25.—Graphs of $y = x^{-1}$, x^{-1} , x^{-2} .

Curves of similar character are obtained by changing the ordinate in any ratio. If the ordinate

is multiplied by c , then all these curves are included in the general formula

$$y = cx^n$$

where c and n may have any values positive or negative. When c is negative, the curves are on the opposite side of the x -axis.

These curves are of importance in connexion with gas and vapour pressures.

§ 11. **Linear Laws.**—These are the laws that contain no fractions involving the variable, and no powers higher than the first: the general form is

$$y = mx + c$$

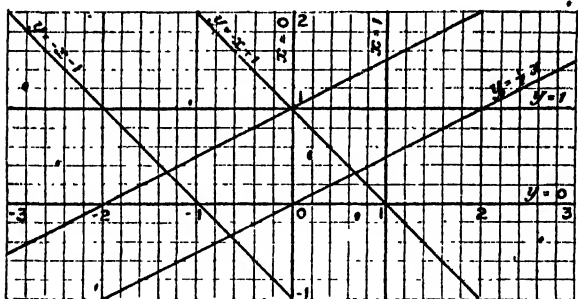


FIG. 26.—Graphs of $y = 0$, $x = 0$, $y = 1$, $x = 1$, $y = \frac{1}{2}x$; $y = -x + 1$; $y = -x - 1$.

where m and c have any values whatever positive or negative.

All the graphs are *straight lines* and they represent quantities that are increasing or decreasing at a constant rate. The straight line represented by

$$y = \frac{1}{2}x$$

passes through the origin and its slope is $\frac{1}{2}$. The graph of

$$y = \frac{1}{2}x + 2$$

is the same straight line as before, but raised through a height of 2 units. In general m is the slope of the straight line, and c is the distance from the origin

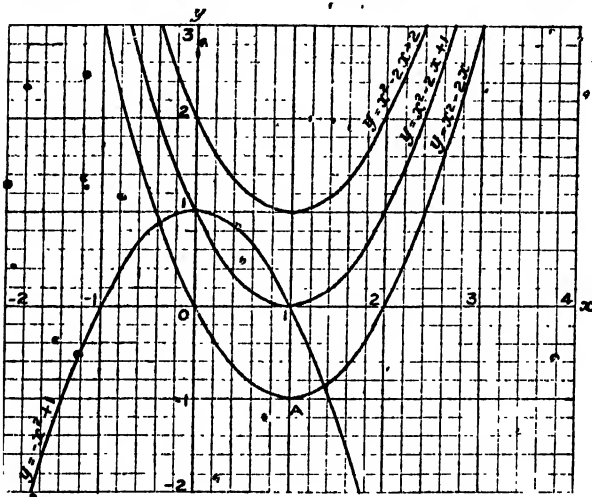


FIG. 27.—Graphs of $y = x^2 - 2x$, $x^2 - 2x + 1$, $x^2 - 2x + 2$, $y = x^2 + 1$.

where it cuts the y -axis. If m is negative the straight line slopes downwards. If c is negative it cuts the y -axis below the origin. The equations of the axes themselves are

$$y = 0 \text{ (the } x\text{-axis) and } x = 0 \text{ (the } y\text{-axis).}$$

It will be noticed that a straight line parallel to the y -axis has to be represented by an equation in which y does not appear, for instance,

$$x = 1 \text{ (in the figure).}$$

§ 12.—Quadratic Laws.—These are the laws con-

taining no power of x higher than the second, and no fractions in x ; their general form is

$$y = ax^2 + bx + c$$

where a, b, c are any constant numbers, positive or negative. *The graphs are always parabolas with their axes vertical.* The parabola is especially important as it occurs frequently in the work of every engineer.

Taking first the equation

$$y = x^2 - 2x,$$

we note that the graph passes through the origin: this is always the case when no term occurs in the equation independent of both x and y (constant term). When x becomes very large, y becomes large at the same time, for after a time x^2 is incomparably larger than $2x$: hence the curve rises indefinitely to the right. It rises to the left also, for when x is negative x^2 is still positive. It follows that the curve has a minimum, the point A.

If we add a constant (unity) to the right hand side of the equation we obtain $y = x^2 - 2x + 1$, which represents the same curve as before, but raised higher up. The graph of

$$y = x^2 - 2x + 2$$

is again the same curve, two units higher.

When the curve crosses the x -axis, we have $y = 0$, so that on the x -axis for the first curve

$$x^2 - 2x = 0.$$

Hence *the abscissæ of the points where the curve cuts the x -axis are the roots of this equation viz. $x = 0, 2$.* Note that the effect of raising the curve bodily is first to bring the two distinct roots into coincidence, and then to make them disappear.

REVERSED PARABOLAS.—The effect of reversing all the signs on the right hand side is to reverse all the ordinates. A law in which the coefficient of x^2 is negative, for example

$$y = -x^2 + 1,$$

where it is -1 , will be represented by a curve having its ends pointing downwards instead of upwards. It will consequently have a maximum instead of minimum.

SLOPE OF PARABOLA.—An important property of these curves is that their slopes are always increasing (decreasing in the last case) at a constant rate. The slope-curves are therefore straight lines. The first three curves will all have the same slope curve ($y = 2x - 2$), for raising or lowering a graph does not affect the slope. In the case of shear and bending moments for beams, for example, to which we have referred already (p. 19), the shear diagram is the slope curve of the B. M. diagram, and the B. M. diagram is the sum curve of the shear diagram. In the case of a uniformly distributed load, the shear diagram is a straight line; therefore the B. M. diagram will be a parabola.

A parabola has no point of inflexion.

§ 13. **Cubic Laws.**—These have the general form

$$y = ax^3 + bx^2 + cx + d.$$

Taking the typical one

$$y = x^3 - x^2 - x + \frac{1}{2},$$

it will be noted that for very great values of x , either positive or negative, the term x^3 will dominate the others. Now when x is negative, x^3 is also negative, so that the curve will descend rapidly on the left-

hand side, and similarly it will rise rapidly on the right-hand side.

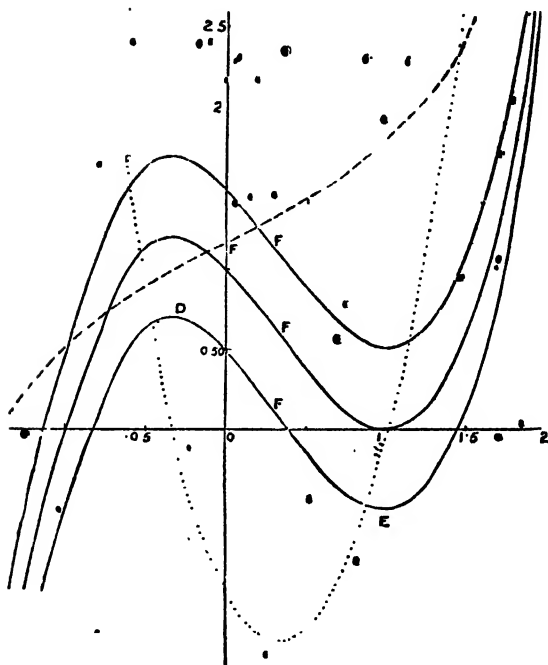


FIG. 28.

$$\left. \begin{aligned} \text{Graphs of } y &= x^3 - x^2 - x + \frac{1}{2} \text{ (lower curve) } \\ &= x^3 - x^2 - x + 1\frac{1}{2} \text{ (centre curve) } \\ &= x^3 - x^2 - x + 2\frac{1}{2} \text{ (upper curve) } \end{aligned} \right\} \text{———}$$

$$y = \frac{x^3}{6} - \frac{x^2}{2} + x + 1 \quad \text{-----}$$

$$\text{(slope curve) } y = 3x^2 - 2x - 1 \quad \text{.....}$$

This curve crosses the axis three times so that the equation

$$x^3 - x^2 - x + \frac{1}{2} = 0$$

has three roots. If the curve is raised, two of these will become coincident and then disappear; two roots (the first two) will also become coincident and disappear if the graph is lowered; but there is always at least one root.

The curve has a maximum D and a minimum E, these being the "critical points". But this is not always the case with cubic curves. The curve

$$y = \frac{x^3}{6} - \frac{x^2}{2} + x + 1$$

has neither a maximum nor a minimum; it will be remarked that the curve bends over as if to form a maximum, but changing its intention, bends back again (F). A cubic cannot have a maximum without having a minimum also.

POINT OF INFLEXION.—A cubic always has one point of inflexion F. The tangent at this point is inclined downwards when there is a maximum and a minimum, and is horizontal or inclined upwards, when there is no maximum or minimum.

SLOPE-CURVE.—The slope-curve of a cubic is a quadratic, i.e. a parabola: it cuts the x -axis at the values of x corresponding to the maximum and minimum when these points exist.

If the coefficient of x^3 is negative, the curve ascends on the left and descends on the right.

QUARTIC AND HIGHER LAWS.—These are of a similar nature to the cubic and quadratic. We will not discuss them in detail, but remark that an $(n-1)$ ic. that is to say a curve whose equation is of the form

$$y = ax^n + bx^{n-1} + cx^{n-2} + \dots$$

cuts the axis at n points at most; it has at most

$n - 1$ critical points, and at most $n - 2$ points of inflexion.

§ 14. **Some other Algebraic Laws.**—The algebraic laws which occur are very varied and it would be impossible to classify them completely. We will content ourselves with giving a few more by way of illustration. When it is required to draw the graph of a given law, it always has to be done ultimately by *finding points* on the curve. But in each case the general shape may be determined, and its drawing very much facilitated by noting its *special features* such as:—

1. The points where it crosses the axes.
2. Its critical points.
3. Its points of inflexion.
- *4. Its asymptotes.
5. The parts which rise and those which fall.
6. Its behaviour for very large positive and negative values of x .
7. Its symmetry about the x -axis, or the y -axis.

These features have all been mentioned already, and some will be discussed in more detail later on.

We will remark here that a curve is symmetrical about the y -axis when only even powers of x occur in its equation, for instance, $y = x^2$, $y = \frac{1}{x^2 - 1}$.

The graph of $y = \frac{1}{x - 1}$ is the same curve as the graph of $y = \frac{1}{x}$ already given in Fig. 25, but it is moved from left to right parallel to the x -axis through a distance of one unit.

The graph of $y = \frac{1}{x^2-1}$ has three asymptotes. The first is the x -axis which the curve approaches indefinitely on the upper side both to the left and to the right. The curve rises indefinitely as we approach $x = -1$ and $x = +1$. These vertical lines

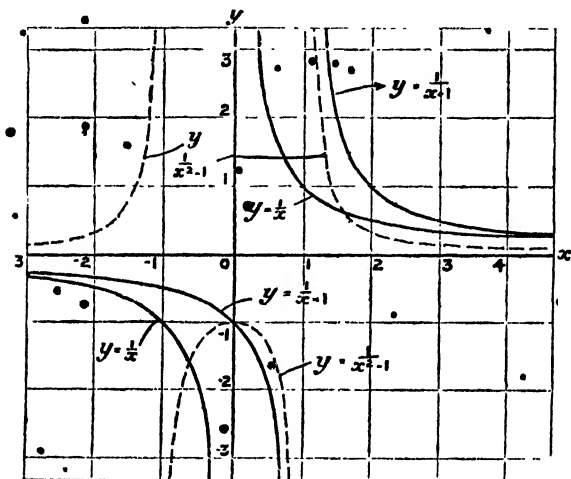


FIG. 29.—Graphs of $y = \frac{1}{x}$, $y = \frac{1}{x-1}$, $y = \frac{1}{x^2-1}$.

are therefore also asymptotes. The graph has one critical point, a maximum $y = -1$ occurring when $x = 0$.

§ 15. **Implicit Laws.**—Very often the dependent variable is given not directly in terms of x (explicitly) as in all the cases considered up to now, but implicitly, that is to say, a general relation is given between x and y . For example we may have

$$x^2 + y^2 = 1 \dots (1)$$

An implicit relation may often be converted into an explicit relation by regarding it as *an equation for determining y* , and solving. For instance the relation

$$x^2 y^2 - 1 = 0$$

when solved becomes

$$y = \frac{1}{x}.$$

If we take

$$\frac{x^2}{4} + y^2 = 1,$$

this may be written

$$y^2 = 1 - \frac{x^2}{4}$$

so that

$$y = + \sqrt{1 - \frac{x^2}{4}} \text{ or } y = - \sqrt{1 - \frac{x^2}{4}} \dots (2)$$

The implicit relation in this case has to be replaced by *two* explicit relations. It is generally more convenient and often necessary to deal with an implicit law as it stands and not to convert it into one or more explicit laws.

The curve (1) is an ellipse (Fig. 30); both axes are axes of symmetry. The *lengths of the semi-axes* are 2 and 1 units respectively. By inspecting (2) it will be seen that the curve cannot exist when $x > 2$. The two equations of (2) give the upper and lower halves of the curve respectively.

The general equation of the ellipse whose semi-axes are a and b is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The curve

$$y^2 = x(x-1)(x+2)$$

cannot exist when x is less than zero, or when x lies between 1 and 2, for such values of x would make y^2 a negative quantity. The curve consists of an

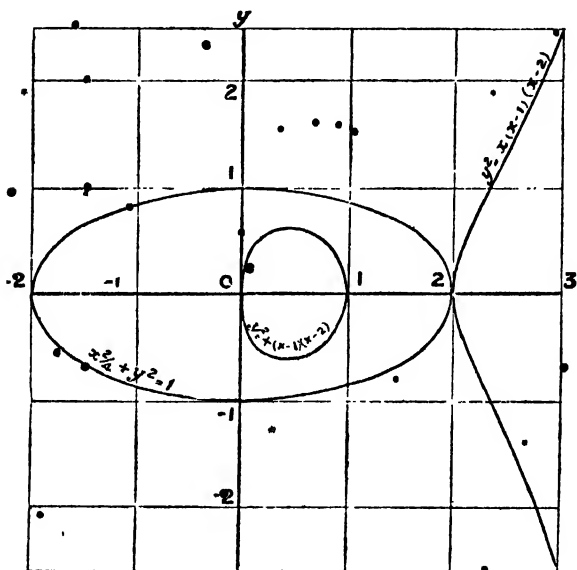


FIG. 30.—Graphs of $\frac{x^2}{4} + y^2 = 1$ and $y^2 = x(x-1)(x-2)$. [Note that the latter has two branches.]

oval and an infinite branch. (Although there is an infinite branch, there are no asymptotes.)

II. TRIGONOMETRIC FORMULÆ.

§ 16.—The Sine Law. Definite laws connecting two variables can be obtained by other means than by algebraical relations; they can for example be

defined geometrically. Trigonometric laws are of this kind.

Expressing the angle $\angle \hat{O}P = x$ in circular measure, so that

$$x = \text{arc } AP/OP$$

and writing

$$y = \sin x = NP/OP,$$

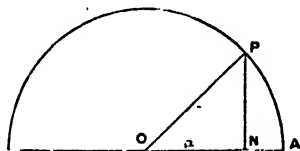


FIG. 31.

we have a law connecting the value of y with that of x .¹

This is a new law, a perfectly definite one, which cannot be expressed algebraically. (See Figs. 15 and 32.)

A very similar law is

$$y = \cos x = ON/OP.$$

The graph of this is in fact not distinct from the sine curve, for remembering that

$$\cos x = \sin \left(x + \frac{\pi}{2} \right),$$

we see that $y = \cos x$ is simply the curve $y = \sin x$,

¹ We wish to emphasize the necessity for taking the correct scale for the sine curve. The abscissæ are the angles in circular measure, the ordinates are the actual values of the sine. The height of the sine curve is unity, and the length of the first arch is $\pi = 3.142$. Although other scales are sometimes convenient in practice, this is the only one giving the true sine law, and the discussion following applies only to this curve.

moved back horizontally through a distance of $\frac{\pi}{2}$ units. (See Figs. 14 and 32.)

§ 17.—**Harmonic Motion.**—This is a good illustration of the sine law.

Suppose a point P to move at a uniform rate (of a radian per second, say) round the circumference of a circle (Fig. 33). Let it start from the point A and let Q be its projection on a vertical KK. At first Q will move at the same rate as P, but its speed will diminish as it rises until it comes to rest at K and then descends. To obtain the displacement curve for this motion, we have only to measure out to the right on the horizontal line through A, the arcs through which P moves: since the speed of P is unity, the abscissæ are the times. We then erect ordinates equal to the ordinates of P. The curve obtained is clearly the sine curve. We shall return to harmonic motion later (see p. 74).

§ 18.—**Properties of the Sine (or Cosine) Law.**—The first property is its *periodicity*. The part of the sine curve which lies between $\theta = 0$ and $\theta = 2\pi$ is repeated between $\theta = 2\pi$ and $\theta = 4\pi$ and again repeated between $\theta = 4\pi$ and $\theta = 6\pi$, and so on (Fig. 32). *The sine curve is periodic, its period being 2π .* This is expressed by the formula

$$\sin(\theta + 2\pi) = \sin \theta.$$

Similarly, of course,

$$\cos(\theta + 2\pi) = \cos \theta.$$

The second property is its *symmetry*. An inspection of the graph will show that the left-hand side of the sine curve is obtained from the right by reflexion first in the y -axis, then in the θ -axis. So that *the sine*

curve is *skew-symmetrical* about the origin (cf. Fig. 24, § 10). The sine-curve is also skew-symmetrical at the points $\theta = \pi, 2\pi, \dots, -\pi, -2\pi, \dots$; it is *symmetrical* about the verticals $\theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}, \theta = -\frac{\pi}{2}$, etc.

The cosine curve similarly is symmetrical about the y -axis, and about the verticals $\theta = \pi, 2\pi, \dots$ while it is skew-symmetrical at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

These facts are expressed by the formulae

$$\sin(-\theta) = -\sin \theta; \cos(-\theta) = \cos \theta, \text{ etc.}$$

which may at once be deduced geometrically from a figure similar to Fig. 31.

§ 19.—**Theorem in Limits.**—A third very important property which may be seen in the graph (Fig. 34) is that the sine curve cuts the axis at an

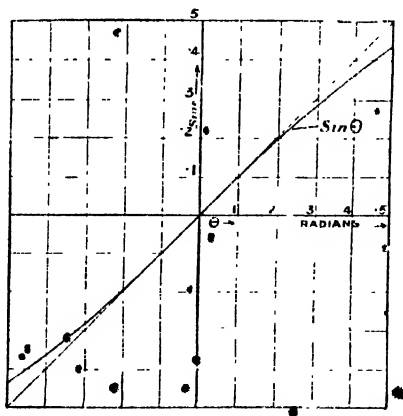


FIG. 34.

angle of 45° $\left(-\frac{\pi}{4}\right)$. This means that near the origin the sine curve

coalesces with the straight line

$$y = \theta,$$

so that when θ is small the ratio

$$\frac{\sin \theta}{\theta} = \frac{\text{ordinate of sine curve}}{\text{ordinate of straight line } y = \theta}$$

approximates to unity. It is very important to remember that θ is in circular measure.

A proof of this important property will be found in textbooks on trigonometry.

§ 20.—The Tangent.—We define the tangent of x as follows :

$$\tan x = \frac{\sin x}{\cos x}.$$

The graph is shown in Fig. 35. Whenever $\cos x$ becomes indefinitely small, $\sin x$ approaches unity, positive or negative, and $\tan x$ therefore becomes indefinitely great. Hence the graph has vertical asymptotes at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

When $\sin x$ approaches zero, $\cos x$ approaches unity. Hence the graph crosses the x -axis at the same points as the sine curve does, and at the same angle, namely 45° . The ratios

$$\frac{\tan x}{\sin x} = \frac{1}{\cos x}$$

and

$$\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

both approach unity when x is small.

The Cosecant is defined by

$$\operatorname{cosec} x = \frac{1}{\sin x},$$

the secant by

$$\sec x = \frac{1}{\cos x},$$

and the cotangent by

$$\cot x = \frac{1}{\tan x}.$$

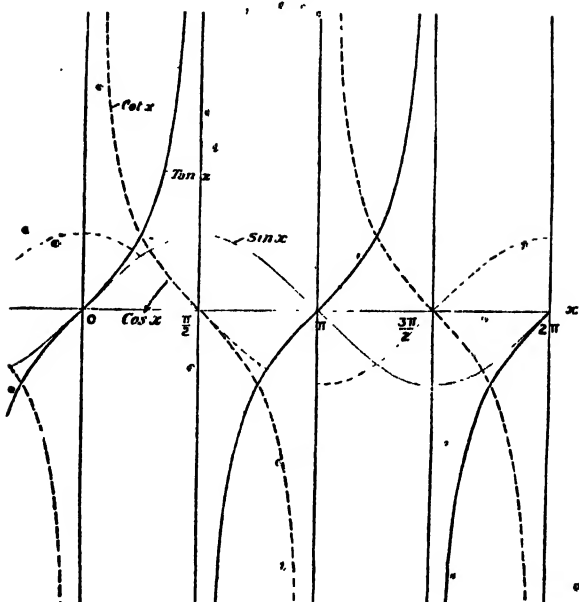


FIG. 35.

The nature of these curves can be inferred easily from the graphs already discussed. They all have vertical asymptotes. The curve $\operatorname{cosec} x$ has maxima where $\sin x$ has minima and minima where $\sin x$ has maxima.

§ 21. **Inverse Relations.**—We can of course regard $\sin x$ as independent and x as depending upon it; given any suitable value of $\sin x = y$ we can find the corresponding value of x . This relation is written

$$x = \sin^{-1} y.$$

$$\text{For } y = 0 \quad x = 0^\circ = 0 \text{ radians}$$

$$y = \frac{1}{\sqrt{2}} \quad x = 45^\circ = \frac{\pi}{4} \text{ radians}$$

$$y = 1 \quad x = 90^\circ = \frac{\pi}{2} \text{ radians,}$$

and so forth

The notation $\cos^{-1} y$, $\tan^{-1} y$, etc., is used similarly.

Note that although $\sin^2 x$ means $(\sin x)^2$, yet $\sin^{-1} x$ does not mean $(\sin x)^{-1} = 1/\sin x$. This confusing notation is unfortunate, but is firmly established. On the Continent $\sin^{-1} x$ is written *arcsin* x .

EXERCISES 2.

1. Draw the graph of the function

$$y = 2x - 9x^2 - 12x + 24.$$

Find for what values of x this function has maximum and minimum values and where its point of inflexion occurs.

2. Rankine's formula for the safe working stress in tons per sq. in. upon mild steel columns is of the

$$\text{form } f_p = \frac{6}{1 + \frac{c^2}{6000}}$$

where c is the "buckling factor".

Plot values of f_p against c and determine the point of inflexion.

3. The bending moment at a distance x from one

end of a beam of span l loaded with a uniformly increasing load W is equal to $M = \frac{Wx}{3} - \frac{Wx^2}{3l}$.

Plot the bending moment diagram for this beam, finding the maximum bending moment and the point at which it occurs.

4. The horse-power transmitted by a certain wire rope is given by the formula $H.P. = \frac{(62790 - 3v^2)v}{230230}$, where v is the velocity in feet per second.

Plot H.P. against v and find the velocity for which it is a maximum.

5. Plot the formula $y = x^2 - 4.2x + 2.93$ and find from your plotting the solution to the equation $x^2 - 4.2x + 2.93 = 0$.

6. Find, by plotting, the value of x which will make $3 \sin x + 2 \cos x$ a maximum.

7. Plot the equation $y = x(81 - x^2)$ and find the maximum value of y and the value of x at which it occurs.

8. The area of the segment of a circle whose arc is l in length and subtends an angle θ at the centre is equal to $\frac{1}{2} l^2 \left(\frac{1}{\theta} - \frac{\sin \theta}{\theta^2} \right)$, θ being in radians.

Plot the area against θ and find where it is a maximum.

9. Plot $y = \sin x (1 + \cos x)$ and find where it is a maximum.

10. The cost in pounds for running a steamer a certain journey is $C = \frac{1300}{v} + v$ where v is the speed in knots.

Plot the cost against the speed and find where it is a minimum.

ANSWERS TO EXERCISES.

2.

1. $x' = -\cdot 56$ and $\dot{3}\cdot 56$. Point of inflection at $x = 1\cdot 5$.
 2. $c = 44\cdot 7$. 3. Max. B.M. = $\cdot 128Wl$ at distance
 $\cdot 577l$ from one end. 4. $83\cdot 5$ feet per sec. 5. $3\cdot 32$,
 $\cdot 88$. 6. $56^\circ 18'$. 7. $280\cdot 6$; $x = 5\cdot 2$. 8. $\theta_1 = \pi$.
 9. $x = \frac{\pi}{3}$. 10. $8\cdot 7$ knots.

CHAPTER III.

LAWS EXPRESSED SYMBOLICALLY (CONTINUED).

III. EXPONENTIAL AND LOGARITHMIC LAWS.

§ 22. **The Compound Interest Law.**—Suppose a sum of money (£ y_0 , say) were put out at 5 per cent. compound interest. At the end of each year would be added to the sum one-twentieth of its amount at the beginning of that year. The growth of the amount could be represented by a graph which would rise by a small jump or step at the end of each year (Fig. 36).

In the diagram distances measured along the horizontal axis represent times in years; the ordinate represents the amount and remains constant during each year, but increases suddenly at the end. The *step* increases in height from year to year, *each step being proportional to the corresponding ordinate* (the figure is illustrative only; the actual graph is much flatter).

If now the interest were added continuously instead of in instalments at the ends of years, at a rate always proportional to the ordinate, and such that the total interest per annum were the same as before, we should have a smooth curve drawn through the upper corners of the steps. This curve represents the *compound interest law*, and its characteristic

property is that its rate of increase or slope is proportional to its ordinate. This property may be expressed by the formula

$$\frac{dy}{dx} = ky \quad \dots \quad (1)$$

and all quantities such that their rates of increase are proportional to themselves follow such a law.

We can easily find the amount at the end of

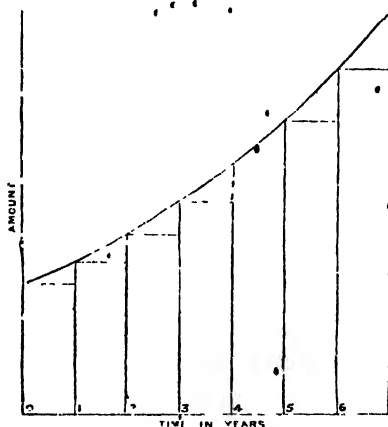


FIG. 36.

each year. If the amount is initially £ y_0 , then at the end of the 1st, 2nd, 3rd, . . . n th . . . year the amount will be $y_0 \cdot 1.05$, $y_0(1.05)^2$, $y_0(1.05)^3$, . . . $y_0(1.05)^n$, . . .

Now the powers to which 1.05 is raised, namely 1, 2, 3, . . . n , . . . are simply the *abscissæ*—if one unit is made to represent one year. Putting in the intermediate values, i.e. *interpolating*, we see that the ordinate of the smooth curve is

$$y = y_0 (1.05)^x$$

where x is *any* abscissa (and not necessarily a whole number).

The general law, when the rate of interest is anything we please, is

$$y = y_0 a^x, \quad (2)$$

on putting a for 1.05. This law has the characteristic property (1), but we shall not at present find the connexion between the number a and the ratio of proportionality k . Since the law is represented by a variable power of a number a , it is also called *exponential*, x being the *exponent* of the power.

It might be thought at first sight that this law was an algebraic one, but this is not so, for in this case the power x is variable. In algebraic expressions, such as x^n , the power is constant.

§ 23. Examples of quantities which follow the exponential laws will be familiar to the student. We take some specific cases.

1°. RATE OF GROWTH OF A POPULATION.—If the rate of increase per thousand is constant, the whole population will increase at a rate proportional to itself at any time.

2°. FRICTION BETWEEN BELT AND PULLEY.—Suppose a belt ARQB to pass round a circular pulley O, and to be about to slip in the direction named. Let

T_a = tension in belt at A

T = " " " " R

$T + \delta T$ = " " " " Q

" = pressure per unit length at R.

r = radius of pulley.

θ = angle AÔR.

$\delta\theta$ = angle RÔQ.

μ = coefficient of friction.

The forces acting upon the element RQ of the belt are

- $Pr\delta\theta$ direction of OR (approximately).
 T " tangent at R , backwards.
 $T + \delta T$ " tangent at Q , forwards.
 $\mu Pr\delta\theta$ " tangent at R , backwards (approximately), the frictional force.

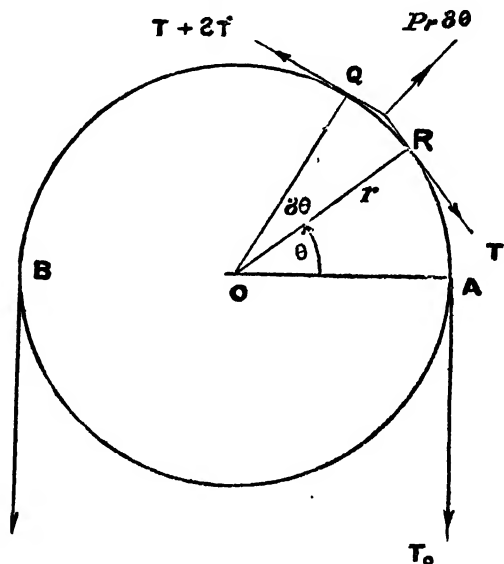


FIG. 37.

Resolving in the direction OR

$$Pr\delta\theta = (T + \delta T) \sin \delta\theta.$$

We may put $\delta\theta$ for $\sin \delta\theta$, and then neglect the product $\delta T \cdot \delta\theta$; thus

$$Pr\delta\theta = T\delta\theta \text{ or } T = Pr \quad (1)$$

Resolving along the tangent at R

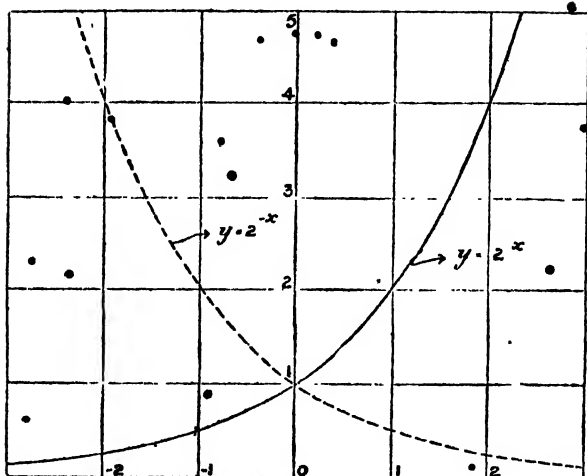
$$T + \mu Pr \delta\theta = (T + \delta T) \cos \delta\theta.$$

Put 1 for $\cos \delta\theta$.

$$T + \mu Pr \delta\theta = T + \delta T,$$

or
$$\delta T = \mu Pr \delta\theta$$

i.e.
$$\frac{\delta T}{\delta\theta} = \mu Pr = \mu T \text{ by (1).}$$



$$\text{--- } y = 2^x \quad \text{---- } y = 2^{-x} = \left(\frac{1}{2}\right)^x$$

FIG. 38.--- $y = a^x$ ($a < 1$) or $y = b^{-x}$ ($b > 1$)

(y_0 is taken as 1 in this figure).

Thus the rate of increase of the tension with the angle is proportional to the tension, which must therefore increase round the pulley exponentially

$$T = T_0 a^{\theta}$$

for a suitable value of a .

We shall mention some other cases later.

* The formula is precisely $T = T_0 e^{\mu\theta}$ (see below).

§ 24. **Decreasing Quantities.**—A quantity also follows an exponential law when its *rate of decrease* is proportional to its value at any time. In fact a rate of decrease is only a *negative* rate of increase. But in this case the number a will be less than unity. We have thus for such a quantity

$$y = y_0 a^x \quad a < 1$$

and

$$\frac{dy}{dx} \text{ (negative) } = -ky$$

where k is some positive number.

Another way of looking at this case is to put $a = \frac{1}{b}$, so that $b > 1$. Then

$$y = y_0 a^x = y_0 \left(\frac{1}{b}\right)^x = y_0 \frac{1}{b^x} = y_0 b^{-x}.$$

From this point of view we see that the curve is exactly the same as the rising curve $y = y_0 b^x$, but reflected in the vertical axis (because opposite signs are taken for x).

§ 25. **Further Examples.**—3°. *Fall of electric current in a circuit from which the E.M.F. is suddenly cut off.* When an electric current C is flowing round a closed circuit, a magnetic field is set up, the number N of lines of magnetic force linking the circuit being proportional to the current. Again if the number N of lines of magnetic force is diminished, then a current is set up in the circuit, its direction being the same as that of the current mentioned in the last sentence, and its magnitude being proportional to the rate of diminution of the number of lines of magnetic force. Therefore in the case in view, the current C at any instant is proportional to

the rate of diminution of N , i.e. to the rate of diminution of C . So

$$C = -\lambda \frac{dC}{dt} \quad (t = \text{time in seconds}).$$

and the current follows a diminishing exponential law.

4°. *Damped Vibrations.*—These will be discussed shortly (p. 75). We mention here that the amplitude A of each vibration is a given fraction, say a (< 1), of the preceding one. Thus successive amplitudes will be

$$A_0, A_0 a, A_0 a^2, A_0 a^3, \dots$$

and in general

$$A = A_0 a^x \quad a < 1$$

(x being here a whole number, viz. the number of vibrations) so that the amplitudes diminish exponentially.

§ 26. **Special Exponential Law.**—We have seen that when y follows the law

$$y = y_0 a^x,$$

then its rate of increase is proportional to itself

$$\frac{dy}{dx} = ky \quad (1)$$

It is clear that there will be one special value of a for which $\frac{dy}{dx}$ is not merely proportional to y , but actually equal to it,

$$\frac{dy}{dx} = y.$$

This value of a is of the utmost importance; it is written e and

$$a = e = 2.718281. \dots$$

The number e is not exact or *commensurable*, that is to say it cannot be represented exactly by an ending

decimal or by a fraction, but only approximately: we have given it to one part in a million.'

Taking *any value* of a we could write

$$a = e^{\lambda}$$

where λ is the power to which we must raise e in order to get a (or λ is the logarithm of a to the base e —see next paragraph). Of course λ also will not be a commensurable number in general. Suppose, for instance, a had the value 10, then

$$\lambda = 2.302585 \dots$$

so that

$$10 = (2.718281 \dots)^{2.302585 \dots}$$

to any required degree of approximation. The calculation by which λ is determined will be easy to anyone familiar with logarithms, and will be briefly explained below.

Thus we may always write

$$y = y_0 e^{\lambda x} \quad (3)$$

instead of (2).

It will now be clear that λ is precisely the number k . For from the above the rate of increase of y with respect to λx is y : so the rate of increase of y with respect to x is λy . So that

$$k = \lambda \quad \frac{dy}{dx} = \lambda y = \lambda y_0 e^{\lambda x} \quad (4)$$

(If the student does not see this at once he should repeat it putting a simple value, say 2, for λ .)

Finally then the two equations (2) (p. 65) and (3) are equivalent to each other.

Thus our final formula for the tension in the belt in the example considered above (2°, p. 65) is $T = T_0 e^{\mu \theta}$.

When a is less than 1, the value of λ will be negative. In the example (3°) above the current is

$$C = C_0 e^{-\lambda t}.$$

§ 27. **Logarithmic Laws.**—If we consider powers of a number, say 10,

$$y = 10^x,$$

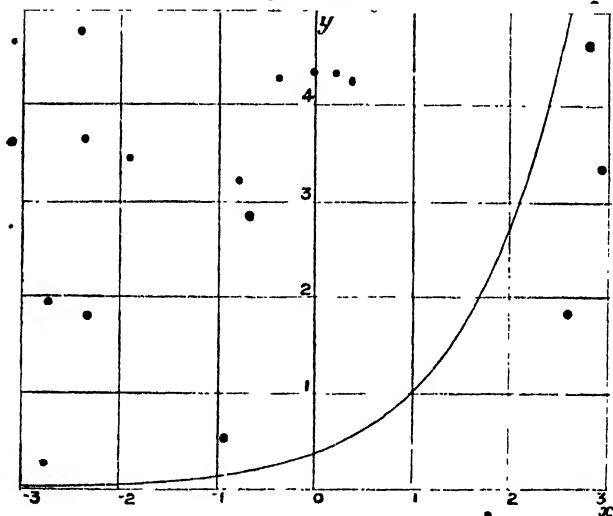


FIG. 39.— $y = 10^x$ ($-y_0 = 1$).

we can, as we have seen, plot them to a curve (similar to Fig. 38) * and we make the following observations:—

The curve is always rising.

When $x = 1$, $y = 10$.

When $x = 0$, $y = 1$.

* This curve is given below (Fig. 40), differently arranged for reasons explained. The student will follow the first part of the reasoning better from Fig. 38, or better still from a figure of his own for $y = 10^x$.

When x increases from 0 to 1, y increases from 1 to 10.

When x is greater than 1 ($x > 1$), y is greater than 10, and y increases indefinitely with x .

When x is negative ($x = -z$), then $y = 10^{-z} = 1/10^z < 1$. The ordinate y becomes indefinitely small, but never vanishes or becomes negative, when x becomes an indefinitely great negative number.

Thus y takes every positive value for some value of x .

Conversely given any value of y , we can always find a power of 10 equal to this value of y ; this is

$x|_2$

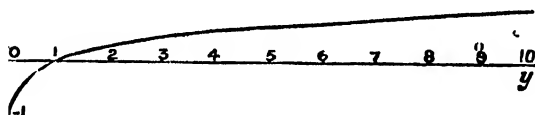


FIG. 10. Graph of $x = \log_{10} y$.

called the *logarithm of y to the base 10*, and is written shortly "

$$\log_{10} y.$$

So that

$$y = 10^x \text{ and } x = \log_{10} y$$

are two ways of writing the same thing.

The graph of $y = 10^x$ will give at once the logarithm of any number: thus to find $\log_{10} 2$, find the ordinate equal to 2 in the figure, and the abscissa will be $\log_{10} 2$. Or we may draw the graph of the logarithm

$$x = \log_{10} y$$

of course making the y (abscissa) axis horizontal, and the x -axis vertical (Fig. 40). But the graph is really the same as before.

Logarithms to base 10 are *common logarithms*; they are used in arithmetical calculations. Logarithms to the base $e = 2.71828 \dots$ are also very important: they are *Napierian* or *Hyperbolic logarithms* and are almost invariably used in theoretical work, for which they are more convenient.

To convert from common logarithms to Napierian logarithms, we use the rule

$$\log_e x = 2.303 \log_{10} x.$$

We have remarked that logarithmic laws and exponential laws are not really distinct: they are *inverse to each other*, just as $y = \sin x$ and $x = \sin^{-1} y$ are inverse to each other. When, as is usual, a law can be expressed in either an exponential or a logarithmic form, the former is preferred. The symbol $\log x$ implies $\log_e x$ in theoretical work and $\log_{10} x$ in numerical work.

The following operation should be noted

$$e^{\log_e x} = x \quad \log_e e^x = x.$$

IV. MIXED LAWS.

§ 28. The Preceding Laws.—

I. $1, x, x^n, \frac{1}{1+x}, \text{etc.}$; . .

II. $\sin x, \cos x, \tan x, \text{etc.}$

$\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \text{etc.}$

III. $e^x, a^x, \log_e x$.

may be combined in a great variety of ways. We will give some important instances.

ADDITION AND SUBTRACTION.—

$$y = x + \sin \frac{1}{2} x.$$

The graph would be obtained by adding the ordinates as in Fig. 41.

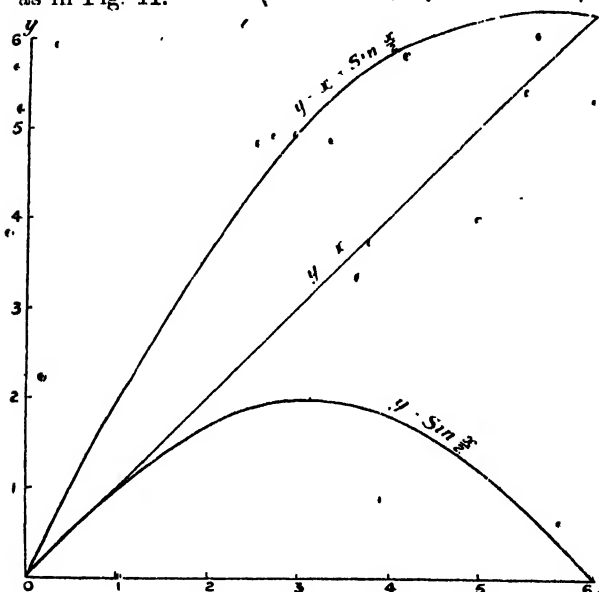


FIG. 41.

MULTIPLICATION AND DIVISION.—Instances are

$$x \sin x; \frac{1}{x} e^x; e^x \log x; \frac{1+x}{\sin x}.$$

The graphs are obtained as before by multiplying or dividing the corresponding ordinates. The following is an example.

DAMPED VIBRATIONS.—This is a very important case; its formula is

$$y = y_0 e^{-\lambda t} \sin \omega t;$$

it is the law followed by all simple *small vibrations whatever, whether mechanical or electrical (see § 25, 4°). The coefficient of $\sin \omega x$, viz. $y_0 e^{-\lambda t}$ is the amplitude; it is seen to diminish exponentially.

“FUNCTION OF A FUNCTION”.—This is a more subtle combination of laws, and one which it is important thoroughly to understand. As we have noted already instead of saying “ y follows such and such a law with respect to x ,” we may say “ y is a function of x ”: thus x^n , $\sin x$ or e^x are all *functions of x* . The general definition is as follows:—

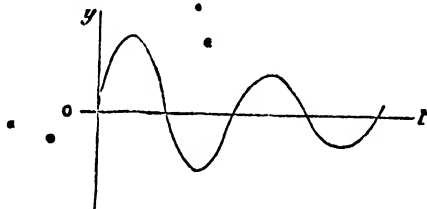


FIG. 42.—Damped vibrations.

A quantity y is a (one-valued) function of x , when to every value of x corresponds one value of y . The statement: “ y is a function of x ” may be written briefly

$$y = f(x).$$

This might mean $y = e^x$, or $y = \sin x$ or that y was given in terms of x empirically by means of a graph, or a table of values.

If y is a function of x (or depends on x), then x is of course a function of y ,

$$x = F(y)$$

* Strictly “with one degree of freedom”. The motion of vibratory systems with several degrees of freedom consists in a combination of several such simple damped vibrations.

since for each value of y there is one (sometimes more than one) value of x . This is the *inverse function*; we use a different letter F , because the law involved is different; this is sometimes written

$$x = f^{-1}(y).$$

If $f(x)$ were e^x , then $f^{-1}(x)$ would be $\log_e x$. We see now the reason for the notation $\sin^{-1} x$, $\cos^{-1} x$, etc.

Now consider

$$y = \sin x^n.$$

Suppose we give a value to x , then x^n takes a definite value, and y is the sine of this last value. To express y in terms of x graphically, we first draw the

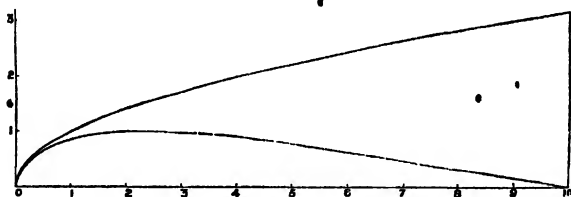


FIG. 43.— $y = x^2$ and $y = \sin x^2$.

graph of x^n (in the figure $y = x^2$), then for the ordinate y corresponding to x , take, not the sine of the number x , but the sine of x^2 , and so on. This is a *function of a function*, for x^n is a function of x , and y is a function of x^n .

The law

$$y = (\sin x)^n = \sin^n x$$

is of course different. We first take the sine of x , then we raise it to the n th power: $\sin x$ is a function of x , and y is a function of $\sin x$.

The process may be repeated several times, and may be combined with the processes already mentioned; the expressions

$$\tan^{-1}(e^x \tan e^x), \sin\left(\frac{1}{1 + \log x}\right)$$

explain themselves simply enough, though their graphs are a little difficult to draw.

Another way of looking at the matter is to introduce some more letters. Thus in the function

$$y = \sin x''$$

we can put

$$z = x''; \quad y = \sin z;$$

then y is a function of z which is itself a function of x .

An example of a rather complicated function involving this idea which occurs in engineering is Hutton's wind pressure formula $P_\theta = P_v \sin^2 \theta^{1.84 \cos \theta}$ where P_θ is the wind pressure upon a surface inclined at an angle θ to the horizontal and P_v is the pressure upon a vertical surface.

EXERCISES 3.

1. Draw a graph of $\log_{10} \sin x$ from $x = 0$ to $x = 90^\circ$.

2. If $x = \frac{1}{2}(e^y - e^{-y})$, show that

$$y = \log_e \{1 + \sqrt{x^2 + 1}\}.$$

3. Plot the graph up to $x = 2$ of the function

$$y = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{2.3} + \frac{x^5}{2.3 \cdot 4.5} + \dots$$

4. Prove that $\sqrt{e} = 1.6487$.

5. Prove that if $x = \log_e \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$

$$\tan \phi = \frac{1}{2}(e^x - e^{-x}).$$

6. Plot against a time base the damped vibration given by $y = 2e^{-.3t} \sin 2.98t$. (Values of y should be calculated by logarithms.)

7. Plot against a time base the function

$$y = .10t \cdot e^{-.3t}.$$

ANSWERS TO EXERCISES.

3.

2. $2x = e^y + \frac{1}{e^y}$ $\therefore e^{2y} - 2xe^y + 1 = 0$. Solving this as a quadratic for e^y , $e^y = x \pm \sqrt{x^2 + 1}$; taking the positive sign (since there is no logarithm of a negative number) $\therefore y = \log \{x + \sqrt{x^2 + 1}\}$.

5. If $x = \log_e \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right)$, $e^x = \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right)$

$$= \frac{1 + \tan \frac{\phi}{2}}{1 - \tan \frac{\phi}{2}} = \frac{\cos \frac{\phi}{2} + \sin \frac{\phi}{2}}{\cos \frac{\phi}{2} - \sin \frac{\phi}{2}} = (\text{multiplying}$$

top and bottom by $\cos \frac{\phi}{2} + \sin \frac{\phi}{2}$)

$$\frac{1 + 2 \sin \frac{\phi}{2} \cos \frac{\phi}{2}}{\cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2}} = \frac{1 + \sin \phi}{\cos \phi}$$

Similarly $e^{-x} = \frac{1 - \sin \phi}{\cos \phi}$

$$\begin{aligned} \therefore e^x - e^{-x} &= \frac{1 + \sin \phi}{\cos \phi} - \frac{1 - \sin \phi}{\cos \phi} \\ &= \frac{2 \sin \phi}{\cos \phi} = 2 \tan \phi. \end{aligned}$$

CHAPTER IV.

SYSTEMATIC DIFFERENTIATION.

§ 29. Slopes obtained from Formulæ.—We have seen in the previous chapters that a relation between two varying quantities, or a *law* can be expressed (1) by means of a graph, (2) by means of a formula. The latter method is not only more terse and convenient from many points of view, but it generally throws a great deal of light upon the connexion between two quantities; and this is especially the case when the law is a simple one. In Chapters II and III we summarized the types of formulæ which are most commonly met with. When we have a quantity expressed as a graph, we can find its rate of change graphically, although this is a tedious and rather inaccurate process. We now want to see how when we are given the quantity as a formula we can find the rate of change or differential coefficient in a second formula.

First we will take some simple examples.

1°. $y = x^2.$

To obtain the slope $\frac{dy}{dx}$, let us find the change in y when x changes by δx :

for x we have $y = x^2$

for $x + \delta x$ we have $y + \delta y = (x + \delta x)^2.$

(79),

The change in $y = \delta y = \text{new value of } y \text{ minus old value of } y$

$$\begin{aligned} &= (x + \delta x)^2 - x^2 \\ &= x^2 + 2x\delta x + \delta x^2 - x^2 \\ &= 2x\delta x + \delta x^2 \end{aligned}$$

\therefore rate of change of y

$$\begin{aligned} &= \text{change in } y \div \text{change in } x \\ &= \delta y / \delta x = [2x\delta x + \delta x^2] / \delta x \\ &= 2x + \delta x. \end{aligned}$$

When δx is made indefinitely small the second term in the last expression becomes indefinitely small, and passes the limits of accuracy, so that it can be neglected.¹ Hence *in the limit*

$$\frac{dy}{dx} = 2x.$$

We have thus found a formula for the slope.

$$2^\circ. \quad y = \frac{1}{1+x}$$

$$\text{Here } \delta y = \frac{1}{1+x+\delta x} - \frac{1}{1+x}$$

(bringing to a common denominator)

¹ Some students find difficulty in following this argument with reference to terms that become so small that they may be neglected. The point to remember is that δx may be as small as

we like. Suppose, for instance $x = 10$. If we take $\delta x = \frac{1}{1000}$,

$\frac{\delta y}{\delta x} = 20 + \frac{1}{1000}$. If it does not satisfy us to neglect this $\frac{1}{1000}$,

take $\delta x = \frac{1}{1,000,000}$; then $\frac{\delta y}{\delta x} = 20 + \frac{1}{1,000,000}$. Thus, getting

smaller and smaller values of δx , we have finally $\frac{dy}{dx} = 20$, i.e.

$$\frac{dy}{dx} = 2x.$$

$$\begin{aligned}
 &= \frac{(1+x) - (1+x+\delta x)}{(1+x+\delta x)(1+x)} \\
 &= \frac{1+x-1-x-\delta x}{(1+x+\delta x)(1+x)} \\
 &= \frac{-\delta x}{(1+x+\delta x)(1+x)}
 \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{(1+x+\delta x)(1+x)},$$

and making δx indefinitely small,

$$\frac{dy}{dx} = \frac{-1}{(1+x)(1+x)} = \frac{-1}{(1+x)^2}.$$

We might differentiate all expressions in this way "from first principles"; it is, however, possible to reduce differentiation to a set of simple rules, and this we shall now do. The method consists in committing to memory a certain number of *standard slopes* (obtained in §§ 31, 32) and deducing all other slopes from these by means of the formulæ in the next article.

§ 30. **General Formulæ.**—We now show how to obtain the slope for the sum (or difference), the product or the quotient, of two quantities, when we know the slopes of these quantities separately.

1°. **SUM:** if $y = y_1 + y_2$

$$\text{then } \frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}.$$

For we have $\delta y = \delta y_1 + \delta y_2$;
dividing by δx ,

$$\frac{\delta y}{\delta x} = \frac{\delta y_1}{\delta x} + \frac{\delta y_2}{\delta x},$$

or making δx (and $\therefore \delta y$) indefinitely small,

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}.$$

Similarly if $y = y_1 - y_2$,

$$\text{then } \frac{dy}{dx} = \frac{dy_1}{dx} - \frac{dy_2}{dx}.$$

2°. PRODUCT: if $y = y_1 y_2$,

$$\text{then } \frac{dy}{dx} = y_2 \frac{dy_1}{dx} + y_1 \frac{dy_2}{dx},$$

or in words: *the slope of* $(y_1 \times y_2) = y_2 \times \text{slope of } y_1 + y_1 \times \text{slope of } y_2$.

Here

$$y + \delta y = (y_1 + \delta y_1)(y_2 + \delta y_2).$$

$$\begin{aligned} \therefore \delta y &= (y_1 + \delta y_1)(y_2 + \delta y_2) - y_1 y_2 \\ &= y_1 y_2 + y_2 \delta y_1 + y_1 \delta y_2 + \delta y_1 \delta y_2 - y_1 y_2 \\ &= y_2 \delta y_1 + y_1 \delta y_2 + \delta y_1 \delta y_2. \end{aligned}$$

Hence

$$\frac{\delta y}{\delta x} = y_2 \frac{\delta y_1}{\delta x} + y_1 \frac{\delta y_2}{\delta x} + \frac{\delta y_1}{\delta x} \delta y_2.$$

If we make δx indefinitely small, the last term becomes negligible, for $\frac{\delta y_1}{\delta x}$ remains finite while δy_2 becomes very small. So finally

$$\frac{d(y_1 y_2)}{dx} = y_2 \frac{dy_1}{dx} + y_1 \frac{dy_2}{dx}.$$

3°. QUOTIENT; if $y = y_2/y_1$

$$\text{then } \frac{dy}{dx} = \left(y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} \right) / y_1^2,$$

Or in words: *the slope of a quotient = denominator \times slope of numerator - numerator \times slope of denominator, all over denominator squared.*

If $y = y_2/y_1$ then

$$\begin{aligned} \delta y &= \frac{y_2 + \delta y_2}{y_1 + \delta y_1} - \frac{y_2}{y_1} \\ &= \frac{(y_2 + \delta y_2)y_1 - y_2(y_1 + \delta y_1)}{(y_1 + \delta y_1)y_1} \end{aligned}$$

$$= \frac{y_1 \delta y_2 - y_2 \delta y_1}{y_1 (y_1 + \delta y_1)}$$

$$\text{Hence } \frac{\delta y}{\delta x} = \frac{y_1 \frac{\delta y_2}{\delta x} - y_2 \frac{\delta y_1}{\delta x}}{y_1 (y_1 + \delta y_1)}$$

$$\text{and } \therefore \frac{dy}{dx} = \frac{y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}}{y_1^2} \text{ because } \delta y_1 \text{ becomes negligibly small.}$$

$$4^\circ. \text{ FUNCTION OF A FUNCTION: } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

or in words: *slope of y with respect to x = slope of y with respect to z \times slope of z with respect to x .*

We now come to the case of the more complicated relations between y and x discussed in § 28 (Chap. III).

Suppose y is given in terms of z so that we know $\frac{dy}{dz}$,

and z is given in terms of x , so that we know also $\frac{dz}{dx}$.

We want to find $\frac{dy}{dx}$. Dealing with finite changes in value or increments we have on introducing δz in numerator and denominator,

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x},$$

when the increments become infinitely small, each ratio becomes a differential coefficient and the formula follows at once, i.e.

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

A special case should be carefully noted. If y depends on x , then x depends on y (inverse function, § 27). So, replacing z by x , instead of saying: y is a function of z which is a function of x , we might say:

x is a function of y , which is a function of x . Therefore

$$\frac{dx}{dy} = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

Now $\frac{dx}{dx}$ is 1; hence

$$1 = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

or,

$$\frac{dx}{dy} = 1 / \frac{dy}{dx},$$

a very useful result.

This can be seen very clearly from the graphical consideration of slopes. If a tangent to a curve intersect the y and x axes at angles α and β respectively, such angles are complementary

$$\therefore \tan \alpha = \cot \beta = 1 / \tan \beta$$

$$\text{but } \tan \alpha = \frac{dx}{dy} \text{ and } \tan \beta = \frac{dy}{dx}$$

§ 31. **Fundamental Slopes.**—We shall reduce the slopes we obtain from first principles (apart from the explanatory examples in § 29) to a minimum, namely three, one for each of the classes of laws in the last chapter.

1°. $y = x^n$, where $n =$ a whole number.

$$\frac{dy}{dx} = n \times x^{n-1}.$$

To prove this we use the *method of induction*.

When $n = 1$, $y = x$ and $\delta y = \delta x$

$$\therefore \frac{\delta y}{\delta x} = 1 \text{ and } \frac{dy}{dx} = 1$$

$$n = 2, \quad y = x^2$$

$$\delta y = (x + \delta x)^2 - x^2 = 2x\delta x + (\delta x)^2$$

$$\therefore \frac{\delta y}{\delta x} = 2x + \delta x \text{ and } \frac{dy}{dx} = 2x + 0 = 2x$$

$$n = 3, y = x^3$$

we have seen in § 29 that $\frac{dy}{dx} = 3x^2$.

Assume the rule holds for any other whole number n . We shall prove that it holds for the next whole number $n + 1$. We assume, then, that

$$\text{if } y = x^n, \text{ then } \frac{dy}{dx} = nx^{n-1}.$$

$$\text{Take } y = x^{n+1} = x^n \times x$$

$$\frac{dy}{dx} = x \frac{dx^n}{dx} + x^n \frac{dx}{dx} \text{ (by § 30)}$$

$$= x \cdot nx^{n-1} + x^n = nx^n + x^n$$

$$= (n + 1) x^n.$$

Then, if the rule holds for any whole number, it holds for the next. It holds for $n = 2$, \therefore for $n = 3$, \therefore for $n = 4$ and so on, and finally it holds for all whole numbers.

$$2^\circ. y = \sin x, \quad \frac{dy}{dx} = \cos x$$

$$y + \delta y = \sin (x + \delta x)$$

$$\delta y = \sin (x + \delta x) - \sin x$$

$$= 2 \sin \frac{\delta x}{2} \cos \left(x + \frac{\delta x}{2} \right).$$

Hence

$$\frac{\delta y}{\delta x} = \frac{2 \sin \frac{\delta x}{2}}{\delta x} \cdot \cos \left(x + \frac{\delta x}{2} \right)$$

$$= \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \cdot \cos\left(x + \frac{\delta x}{2}\right).$$

The first factor $\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$ approaches unity when

$\frac{\delta x}{2}$ becomes indefinitely small (see § 19); the second factor approaches $\cos x$. Hence

$$\frac{dy}{dx} = \cos x.$$

$$3^\circ. y = e^x, \quad \frac{dy}{dx} = e^x.$$

This we have seen is a characteristic property of the exponential law (§ 26).

§ 32. **Standard Slopes.**—We now deduce a set of standard slopes from the results in §§ 26 and 27.

$$1^\circ. y = x^n \text{ (} n = \text{any number)}, \quad \frac{dy}{dx} = nx^{n-1}$$

(1) let n be fractional: $n = p/q$

$$y = x^{\frac{p}{q}}.$$

Raise both sides to the q^{th} power

$$y^q = x^p.$$

Differentiate both sides with regard to x .

The left-hand side y^q is a function of y , which itself is a function of x ; we therefore apply the "function of a function," rule (§ 30, 4^c), and get

$$\frac{dy^q}{dy} \cdot \frac{dy}{dx} = \frac{dx^p}{dx}$$

$$4. \quad qy^{q-1} \cdot \frac{dy}{dx} = px^{p-1}.$$

$$\begin{aligned} \text{So } \frac{dy}{dx} &= \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{\left(\frac{p}{qx^q}\right)^{q-1}} \\ &= \frac{p}{q} \frac{x^{p-1}}{x^{p-\frac{p}{q}}} = \frac{p}{q} x^{p-1-\frac{p}{q}} \\ &= \frac{p}{q} x^{\frac{p}{q}-1} = \frac{p}{q} x^{n-1}. \end{aligned}$$

(2) Let n be negative: $n = -m$ (m is a positive number, whole or fractional)

$$\begin{aligned} y &= x^{-m} \\ \therefore x^m y &= 1. \end{aligned}$$

Differentiate both sides: the slope of 1 is zero, for the graph of a constant is a horizontal straight line: the left-hand side is a product (§ 30, 2°).

$$\begin{aligned} y \frac{dx^m}{dx} + x^m \frac{dy}{dx} &= 0 \\ \text{i.e. } x^m \frac{dy}{dx} &= -y \frac{dx^m}{dx} = -x^{-m} \cdot mx^{m-1} \\ &= -mx^{-1} \\ \frac{dy}{dx} &= -mx^{-2} \end{aligned}$$

The result then holds for all values of n .

$$2^\circ. \quad y = \cos x, \quad \frac{dy}{dx} = -\sin x.$$

We can write $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, and apply the "function of a function" rule (§ 30, 4°), for $\sin\left(x + \frac{\pi}{2}\right)$ is a function of $\left(x + \frac{\pi}{2}\right)$ which is a function of x .

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d \cos x}{dx} = \frac{d \sin \left(x + \frac{\pi}{2} \right)}{dx} \\
 &= \frac{d \sin \left(x + \frac{\pi}{2} \right)}{d \left(x + \frac{\pi}{2} \right)} \cdot \frac{d \left(x + \frac{\pi}{2} \right)}{dx} \\
 &= \cos \left(x + \frac{\pi}{2} \right) \cdot \left(\frac{dx}{dx} + \frac{d \frac{\pi}{2}}{dx} \right) \\
 &= -\sin x \cdot (1 + 0) = -\sin x
 \end{aligned}$$

$$3^\circ. y = \tan x, \frac{dy}{dx} = \sec^2 x.$$

Here $\tan x = \frac{\sin x}{\cos x}$ is to be regarded as a quotient (§ 30, 3°)

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) = \frac{\cos x \frac{d \sin x}{dx} - \sin x \frac{d \cos x}{dx}}{\cos^2 x} \\
 &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.
 \end{aligned}$$

$$4^\circ. y = \sin^{-1} x, \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Here we have

$$\sin y = x.$$

Applying the "function of a function" rule (§ 30, 4°)

$$\begin{aligned}
 \frac{\sin y}{dy} \cdot \frac{dy}{dx} &= \frac{dx}{dx} \\
 \cos y \frac{dy}{dx} &= 1
 \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

$$5^\circ. y = \tan^{-1} x, \frac{dy}{dx} = \frac{1}{1 + x^2}$$

Proceeding as in 4°, we put $\tan y = x$,

$$\therefore \sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$6^\circ. y = \log x, \frac{dy}{dx} = \frac{1}{x}$$

Again similarly

$$\begin{aligned} & e^y = x \\ \therefore \frac{de^y}{dy} \cdot \frac{dy}{dx} &= \frac{dx}{dx} \\ e^y \frac{dy}{dx} &= 1 \\ \therefore \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x} \end{aligned}$$

7°. The student should notice the effect of *multiplying by a constant*. We will illustrate this by means of examples:—

$$(1) y = a \sin x.$$

It should be evident that

$$\frac{dy}{dx} = a \frac{d \sin x}{dx} = a \cos x.$$

If it is not evident, the product formula (§ 30, 2°) may be used

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} a \cdot \sin x \\ &= \sin x \frac{da}{dx} + a \frac{d \sin x}{dx} \\ &= \sin x \times 0 + a \cos x \end{aligned}$$

$$= a \cos x.$$

$$(2) \quad y = \sin bx.$$

Regard this as a "function of a function".

$$\begin{aligned} \frac{dy}{dx} &= \frac{d \sin bx}{dx} = \frac{d \sin (bx)}{d(bx)} \cdot \frac{d(bx)}{dx} \\ &= \cos bx \times b = b \cos bx. \end{aligned}$$

The following list of standard slopes should be remembered. The student should work out for himself those which have not been obtained in the text.

LIST OF STANDARD SLOPES OR DIFFERENTIAL COEFFICIENTS.

| | | |
|-----|--------------------------|------------------------------------|
| 1. | $y = \text{constant}$ | $\frac{dy}{dx} = 0$ |
| 2. | $y = x^n$ | $\frac{dy}{dx} = nx^{n-1}$ |
| 3. | $\sin x$ | $\cos x$ |
| 4. | $\cos x$ | $-\sin x$ |
| 5. | $\tan x$ | $\sec^2 x$ |
| 6. | $\operatorname{cosec} x$ | $-\operatorname{cosec}^2 x \cos x$ |
| 7. | $\sec x$ | $\sec^2 x \sin x$ |
| 8. | $\cot x$ | $-\operatorname{cosec}^2 x$ |
| 9. | $\sin^{-1} x$ | $1/\sqrt{1-x^2}$ |
| 10. | $\cos^{-1} x$ | $-1/\sqrt{1-x^2}$ |
| 11. | $\tan^{-1} x$ | $1/(1+x^2)$ |
| 12. | e^x | e^x |
| 13. | $\log x$ | $1/x$ |

§ 33. **Implicit Relations.**—We have so far considered cases where the dependent quantity y is given directly (*explicitly*) in terms of x , as in

$$y = \frac{1}{1+x}.$$

We frequently have y given in terms of x in an indirect (*implicit*) way, as in

$$1^\circ. x^2 + 2xy + 1 + y = 0.$$

$$2^\circ. y^2 + 3xy + 2x^2 = 0.$$

$$3^\circ. e^{xy} = x + y.$$

As before y can be determined when x is known, but only by solving an equation. In 1° we get the explicit form at once . . .

$$y = -\frac{x^2 + 1}{2x + 1}$$

In 2° we find (on solving the quadratic equation in y) our relation is equivalent to *two explicit relations*

$$y = -x \text{ and } y = -2x.$$

In 3° we *cannot find* the equivalent explicit relations.

In such cases it is still often necessary and easy to find $\frac{dy}{dx}$. To do so we only have to differentiate the equation through, remembering that y depends upon x (is a function of x).

Taking 2°

$$2y \frac{dy}{dx} + 3y + 3x \frac{dy}{dx} + 4x = 0.$$

$$\text{Hence} \quad \frac{dy}{dx} = -\frac{3y + 4x}{2y + 3x}.$$

It is seen that in this case both x and y appear in the expression for the slope. This as a rule is no disadvantage, and even in cases where we can replace y by its expression in x (we could in this example), it is generally not worth while doing so.

In example 3° we have similarly

$$\frac{d(e^{xy})}{d(xy)} \frac{d(xy)}{dx} = \frac{dx}{dx} + \frac{dy}{dx},$$

$$\text{i.e. } e^{xy} \left(y' + x \frac{dy}{dx} \right) = 1 + \frac{dy}{dx}.$$

Hence

$$\frac{dy}{dx} = \frac{1 - ye^{xy}}{xe^{xy} - 1}.$$

§ 34. **Successive Differentiation.**—Having found the slope of any expression, say x^n , we have another expression, nx^{n-1} , of which we can find the slope in turn (§ 5). And this process may be carried out as often as we please. These successive differential coefficients are written

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \text{ etc.}$$

So that

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = \left(\frac{d}{dx} \right) y, \text{ etc.}$$

Thus if

$$1^{\circ} \quad y = x^n,$$

then

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{d^2y}{dx^2} = n(n-1)x^{n-2},$$

$$\frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3},$$

$$\frac{d^4y}{dx^4} = n(n-1)(n-2)(n-3)x^{n-4},$$

and so on.

Note that the n th slope when n is integral,

$$\frac{d^n y}{dx^n} = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1.$$

is a constant, and all higher slopes vanish.

Other examples :—

2°. $y = \log x$

$$\frac{dy}{dx} = \frac{1}{x}, \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}, \quad \frac{d^3y}{dx^3} = +\frac{2}{x^3},$$

$$\frac{d^4y}{dx^4} = -\frac{2 \cdot 3}{x^4} = -\frac{6}{x^4}, \text{ and so on.}$$

3°.

$$\frac{dy}{dx} = \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = \dots = e^x = y.$$

4°.

$$y = \sin x$$

$$\frac{dy}{dx} = \cos x = \sin \left(x + \frac{\pi}{2} \right)$$

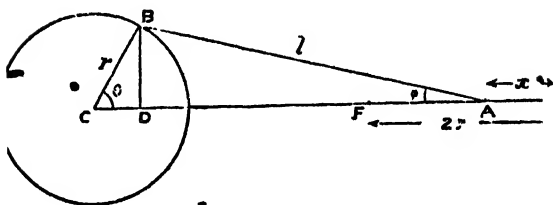


FIG. 44.

$$\frac{d^2y}{dx^2} = \cos \left(x + \frac{\pi}{2} \right) = -\sin x$$

$$\frac{d^3y}{dx^3} = \sin \left(x + 3 \cdot \frac{\pi}{2} \right) = -\cos x$$

$$\frac{d^4y}{dx^4} = \sin \left(x + 4 \cdot \frac{\pi}{2} \right) = \sin x.$$

§ 35. Example of Steam-Engine Mechanism.—
As a useful example involving fairly complicated differentiation, take the case of the steam engine mechanism that we have already considered (p. 13).

The crank CB revolves around the point C with uniform velocity and the cross-head A reciprocates

between the points BE. To find the velocity and acceleration of the cross-head.

Let the cross-head be at a distance x from E.

Velocity.—Now velocity = rate of change of position, i.e. $\frac{dx}{dt}$

$$x = EC - AC$$

$$= l + r - l \cos \phi - r \cos \theta.$$

$$\text{Let } l = kr,$$

$$\text{then } x = r \{k(1 - \cos \phi) + (1 - \cos \theta)\} \quad (1)$$

$$\therefore v = \frac{dx}{dt}$$

$$= r \left\{ k \frac{d(1 - \cos \phi)}{d\phi} \cdot \frac{d\phi}{dt} + \frac{d(1 - \cos \theta)}{d\theta} \cdot \frac{d\theta}{dt} \right\}^*$$

$$= r \left\{ k \sin \phi \frac{d\phi}{dt} + \sin \theta \frac{d\theta}{dt} \right\} \quad (2)$$

Now $\frac{d\theta}{dt}$ = rate of change of angle = angular velocity = $\omega = 2\pi n$, n being the number of revolutions per second.

$$\text{Also } l \sin \phi = BD = r \sin \theta,$$

$$\therefore k \sin \phi = \sin \theta, \quad (3)$$

\therefore differentiating we get

$$k \cos \phi \frac{d\phi}{dt} = \cos \theta \frac{d\theta}{dt} \quad (4)$$

Substituting (3) and (4) in (2),

$$v = r \left(\frac{k \sin \phi \cos \theta}{k \cos \phi} \cdot \frac{d\theta}{dt} + \sin \theta \frac{d\theta}{dt} \right)$$

* This is an example of "Function of a function" (pp. 75 and 83).

$$r \frac{d\theta}{dt} \left(\frac{\sin \theta \cos \theta}{k \cos \phi} + \sin \theta \right) \\ = 2 \pi r n \sin \theta \left(1 + \frac{\cos \theta}{k \cos \phi} \right) \quad (5)$$

$$\text{From (3)} \quad k^2 \sin^2 \phi = \sin^2 \theta$$

$$\text{i.e.} \quad k^2 (1 - \cos^2 \phi) = \sin^2 \theta$$

$$\text{i.e.} \quad k^2 \cos^2 \phi = k^2 - \sin^2 \theta$$

$$\text{or } k \cos \phi = \sqrt{k^2 - \sin^2 \theta}$$

$$\therefore v = 2 \pi r n \sin \theta \left(\frac{\cos \theta}{\sqrt{k^2 - \sin^2 \theta}} + 1 \right) \quad (6)$$

Acceleration.—Now acceleration = rate of change of velocity

$$= a = \frac{dv}{dt} = \frac{dv}{d\theta} \cdot \frac{d\theta}{dt} = 2 \pi n \cdot \frac{dv}{d\theta} \quad (7)$$

$$\therefore a = 4 \pi^2 n^2 r \sin \theta \cdot \frac{d}{d\theta} \left(\frac{\cos \theta}{\sqrt{k^2 - \sin^2 \theta}} + 1 \right)$$

$$+ \left(\frac{\cos \theta}{\sqrt{k^2 - \sin^2 \theta}} + 1 \right) \frac{d \sin \theta}{d\theta} \quad (8)$$

$$\text{Now} \quad \frac{d}{d\theta} \left(\frac{\cos \theta}{\sqrt{k^2 - \sin^2 \theta}} + 1 \right)$$

$$\frac{\sqrt{k^2 - \sin^2 \theta} \cdot \frac{d \cos \theta}{d\theta} - \cos \theta \cdot \frac{d \sqrt{k^2 - \sin^2 \theta}}{d\theta}}{(k^2 - \sin^2 \theta)} + 0$$

$$= \frac{-\sin \theta \sqrt{k^2 - \sin^2 \theta} - \cos \theta \cdot \frac{1}{2} (\sqrt{k^2 - \sin^2 \theta})^{-\frac{1}{2}} (-2 \sin \theta \cos \theta)}{(k^2 - \sin^2 \theta)}$$

$$\frac{\sin \theta \cos^2 \theta}{\sqrt{k^2 - \sin^2 \theta}} - \sin \theta \sqrt{k^2 - \sin^2 \theta}$$

$$\frac{d \sin^2 \theta}{d\theta} = 2 \sin \theta \frac{d \sin \theta}{d\theta} = 2 \sin \theta \cos \theta$$

$$\begin{aligned}
 &= \frac{\sin \theta \{\cos^2 \theta - (\sqrt{k^2 - \sin^2 \theta})^2\}}{(k^2 - \sin^2 \theta)^{\frac{3}{2}}} \\
 &= \frac{\sin \theta (\cos^2 \theta - k^2 + \sin^2 \theta)}{(k^2 - \sin^2 \theta)^{\frac{3}{2}}} = \frac{\sin \theta (1 - k^2)}{(k^2 - \sin^2 \theta)^{\frac{3}{2}}}
 \end{aligned}$$

Putting this value in (8) we get

$$\begin{aligned}
 a &= 4\pi^2 n^2 r \left\{ \frac{\sin^2 \theta (1 - k^2)}{(k^2 - \sin^2 \theta)^{\frac{3}{2}}} \right. \\
 &\quad \left. + \cos \theta \left(1 + \frac{\cos \theta}{\sqrt{k^2 - \sin^2 \theta}} \right) \right\} \\
 &= 4\pi^2 n^2 r \left\{ \cos \theta + \frac{\sin^2 \theta (1 - k^2) + \cos^2 \theta (k^2 - \sin^2 \theta)}{(k^2 - \sin^2 \theta)^{\frac{3}{2}}} \right\} \\
 &= 4\pi^2 n^2 r \left\{ \cos \theta + \frac{k^2 (\cos^2 \theta - \sin^2 \theta) + \sin^2 \theta (1 - \cos^2 \theta)}{(k^2 - \sin^2 \theta)^{\frac{3}{2}}} \right\} \\
 &= 4\pi^2 n^2 r \left\{ \cos \theta + \frac{k^2 \cos 2\theta + \sin^4 \theta}{(k^2 - \sin^2 \theta)^{\frac{3}{2}}} \right\} \quad (8)
 \end{aligned}$$

Equation (8) therefore gives the acceleration of the cross-head and therefore that of the piston for any angular position of the crank.

EXERCISES 4.

When the student has grasped the principles of differentiation he requires considerable practice in carrying out the operation in definite exercises and so should work carefully through the following:—

Find the differential coefficients with regard to x of:—

1. $5x^2$.

2. $6x^2 - 6x + 12$.

3. $4 + 11x - 2x^2$.

4. $\frac{2}{x^3}$.

5. $\frac{4}{\sqrt{x^5}}$.

6. $+\frac{1}{x}$.

7. $4x + \frac{1}{x^2}$.

8. $(4x + 1)^5$.

9. $(3 - x)^4$.

- | | |
|-----------------------------|-------------------------------------|
| 10. $\frac{1}{(2x+4)^2}$ | 17. $x \sin x$ |
| 11. $\frac{1}{(x+1)^2}$ | 18. $x \log_e x$ |
| 12. $\frac{1}{(x+1)^2}$ | 19. $\log_e \sin x$ |
| 13. $\frac{2x^4}{x^2+1}$ | 20. $\sin^{-1}(1-x)$ |
| 14. $\sin \sqrt{x^2+3}$ | 21. $\log_e \frac{1-x}{1+x}$ |
| 15. $\sin^2 x$ | 22. $\log_e \{x + \sqrt{x^2+1}\}$ |
| 16. $\sin 2x \cos x$ | 23. $\log_e \{x + \sqrt{x^2-1}\}$ |
| 26. $\sin x^2$ | 24. $y = \frac{1}{2}(e^x - e^{-x})$ |
| 27. $\sin^{-1} 2x$ | 25. $\sqrt{\frac{1+x^2}{1-x^2}}$ |
| 28. $\tan^{-1} x^2$ | 30. $2 \log_{10} x$ |
| 29. $\cos^{-1} \frac{1}{x}$ | 31. $\cot \log_e x$ |
| | 32. $\log \cot x$ |
| | 33. e^{-x} |
| | 34. e^{-x} |
| | 35. $e^{-\frac{1}{10}} \sin 3x$ |

36. If $s = a \cos (mt + b)$, find $\frac{d^2s}{dt^2}$.

37. If $y = A \sin mx + B \sin nx$, find $\frac{d^2y}{dx^2}$.

38. If $y = x^3 \log_e \frac{x}{a}$, find $\frac{d^4y}{dx^4}$.

39. If $y = 3x^3 + 4x^2 - 3x + 6$, find $\frac{d^2y}{dx^2}$.

40. If $y = \sin^2 x$, find $\frac{d^2y}{dx^2}$.

41. In the steam-engine, show that when the piston is at its mid point in the forward motion, the crank angle is $\cos^{-1} \frac{1}{2k}$ and find the acceleration and velocity in that position.

ANSWERS TO EXERCISES.

4.

1. $10x$. 2. $12x$. 3. $11 - 4x$. 4. $\frac{-6}{x^4}$
5. $-\frac{10}{\sqrt{x^5}}$. 6. $1 - \frac{1}{x^2}$. 7. $4 - \frac{2}{x^3}$.
8. $20(4x+1)^4$. 9. $\frac{1}{2} 4(3-x)^4$. 10. $-\frac{4}{(2x+4)^3}$
11. $-\frac{2}{(x+1)^3}$. 12. $\frac{2(1-x^2)}{(x^2+1)^2}$. 13. $\frac{128x^3 - 4x^5}{(16-x^2)^2}$.
14. $\frac{x \cos \sqrt{x^2+3}}{\sqrt{x^2+3}}$. 15. $\sin 2x$.
16. $2 \cos 2x \cos x - \sin 2x \sin x$.
17. $\sin x + x \cos x$. 18. $1 + \log x$. 19. $\cot x$.
20. $\frac{-1}{\sqrt{2x-x^2}}$. 21. $\frac{-2}{1-x^2}$. 22. $\frac{1}{\sqrt{1+x^2}}$
23. $\frac{1}{\sqrt{x^2-1}}$. 24. $\frac{1}{2}(e^x + e^{-x})$. 25. $\frac{2x}{\sqrt{(1+x^2)(1-x^2)^3}}$
26. $2x \cos x^2$. 27. $\frac{2}{\sqrt{1-4x^2}}$. 28. $\frac{2x}{(1+x^4)}$
29. $\frac{1}{x \sqrt{x^2-1}}$. 30. $2 \log_{10} e \cdot \frac{1}{x}$.
31. $-\frac{1}{x} \operatorname{cosec}^2 \log x$. 32. $-\frac{1}{\sin x \cos x}$.
33. $-e^{-x}$. 34. $-2xe^{-x^2}$.
35. $e^{-\frac{x}{10}} \left(-\frac{\sin 3x}{10} + 3 \cos 3x \right)$.
36. $-m^2s$. 37. $-m^2y$. 38. $\frac{6}{x}$. 39. $18x + 8$.
40. $2 \cos 2x$. 41. $v = \frac{2\pi nrk \sqrt{4k^2-1}}{(2k^2-1)^{\frac{3}{2}}}$
- $\alpha = \frac{(2\pi n)^2 (4k^2-6k^2+1)}{(2k^2-1)^3}$.

CHAPTER V.

APPLICATIONS OF DIFFERENTIATION.

CRITICAL VALUES—MAXIMA AND MINIMA.

§ 36. Let us return to the discussion in § 3. If the figure represents the graph corresponding to some law, then the point

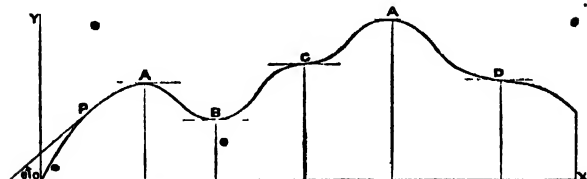


FIG. 45.

A is a maximum, since y rises to a greatest value and then falls ;

B is a minimum, since y falls to a least value and then rises.

At each of these points the curve is for a moment horizontal, and if θ is the inclination of the tangent at any point to the x -axis, we shall have at A and B

$$\text{slope} = \frac{dy}{dx} = \tan \theta = 0.$$

But $\frac{dy}{dx}$ vanishes also at such a point as C where

the curve rises and becomes horizontal only to rise again.

The points

C and D are points of inflexion with horizontal tangents,

and all such points as A, B, C, D are critical points.

We now have to deal with this problem:—

Given a law in the shape of a formula to find its critical points and to discuss their nature.

In the first place all the critical values are obtained by writing

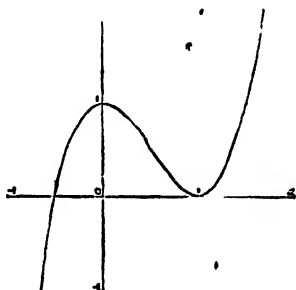


FIG. 46.

$$\frac{dy}{dx} = 0;$$

for example, if

$$y = 2x^3 - 3x^2 + 1, \quad (1)$$

then the critical values are given by

$$\frac{dy}{dx} = 6x^2 - 6x = 0$$

$$\text{i.e. } x^2 - x = 0, \quad x(x-1) = 0$$

$$x = 0 \text{ and } x = 1;$$

the critical values of y are

$$(1) \text{ for } x = 0, y = 0 + 0 + 1 = 1$$

$$(2) \text{ for } x = 1, y = 2.1 - 3.1 + 1 = 0.$$

Having obtained the critical values, it is necessary to find out whether they are maxima or minima or only points of inflexion.

Drawing the derived curve of Fig. 45, we note (Fig. 47) that the slope which was positive before the point A, vanishes at A (M) and becomes negative. So

At a maximum the slope is diminishing, and the slope curve is descending.

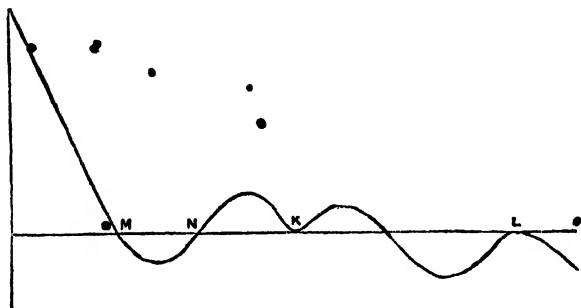


FIG. 47.

The slope of the slope curve is negative, i.e.

$$\frac{d^2y}{dx^2} \text{ is negative at a maximum.}$$

Similarly

$$\frac{d^2y}{dx^2} \text{ is positive at a minimum.}$$

At a point of inflexion such as C (K) (Fig. 45), the slope which was positive diminishes down to zero, but then it *increases* immediately after. We must therefore have a minimum on the slope curve; similar remarks apply to a point such as D (L), and hence

$\frac{d^2y}{dx^2} = 0$ at a point of inflexion (with a horizontal tangent).

For the expression (1)

$$\frac{d^2y}{dx^2} = 12x - 6.$$

Hence

(1) for $x = 0$, $\frac{d^2y}{dx^2} = -6$, maximum.

(2) for $x = 1$, $\frac{d^2y}{dx^2} = 12 - 6 = +6$, minimum.

As an example of a point of inflexion, take

$$y = x(x-1)^3;$$

the critical values are the roots of

$$\frac{dy}{dx} = (x-1)^3 + 3x(x-1)^2 = 0,$$

$$\text{i.e. } (x-1)^2(x-1+3x) = 0$$

$$(x-1)^2(4x-1) = 0$$

$$\therefore x = 1 \text{ and } x = \frac{1}{4}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \{(x-1)^2(4x-1)\}$$

$$= 2(x-1)(4x-1) + 4(x-1)^2$$

$$= 6(x-1)(2x-1)$$

$$x = 1 \text{ gives } \frac{d^2y}{dx^2} = 0, \text{ point of inflexion.}$$

$$x = \frac{1}{4} \text{ gives } \frac{d^2y}{dx^2} = \frac{9}{4}, \text{ minimum.}$$

It should be noted that in general a maximum value of y is not the greatest value, for after rising to a maximum and falling, the curve may rise again beyond the maximum. An inspection of Fig. 45 will show further that maxima and minima must

generally occur alternately (always when there is no break in the curve).

Note.—The above investigation is not complete, although quite exact as far as it goes. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at a certain point, this point is generally a point of inflexion, but it may exceptionally still be a maximum or a minimum. The complete rule is:
At a critical point let

$$\frac{d^2y}{dx^2} = 0, \frac{d^3y}{dx^3} = 0, \text{ etc.}$$

the first slope which does not vanish being

$$\frac{d^ry}{dx^r}$$

then

(1) if r is odd, we have a point of inflexion.

(2) if r is even, then

if $\frac{d^ry}{dx^r} < 0$ we have a maximum,

if $\frac{d^ry}{dx^r} > 0$ we have a minimum.

For instance, consider

$$y = (x - 1)^4$$

$$\frac{dy}{dx} = 4(x - 1)^3: \text{critical value is } x = 1;$$

$$\text{for } x = 1, \frac{d^2y}{dx^2} = 4.3(x - 1)^2 = 0$$

$$\frac{d^3y}{dx^3} = 4.3.2(x - 1) = 0$$

$$\frac{d^4y}{dx^4} = 4.3.2.1 = 24.$$

Hence $x = 1$ is a minimum.

EXERCISES 5A.

Simple Problems on Maxima and Minima.

1. Find the critical value of $8 + 15x - 3x^2$. Is it a maximum or a minimum?

2. Find the maximum and minimum of

$$x^3 - 8x + 6.$$

3. Find the maximum and minimum values of $x(9 - x^2)$.

4. What is the maximum value of $(2x + 3)(5 - x)$?

5. Prove that the rectangle of least perimeter for a given area is the square.

6. Find the area of the greatest rectangle that can be inscribed in a given triangle.

7. Find the values of x for a maximum value of the graph $y = xe^{-x}$.

8. Show that $\sin x(1 + \cos x)$ is a maximum for $x = 60^\circ$.

9. Find the minimum value of $a \cot \theta + b \tan \theta$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

10. Find the values of θ which make $\cos \theta + \sin \theta$ a maximum.

§ 37. **Worked Problems on Maxima and Minima.**—The following examples will show some of the many applications to problems of engineering interest of the above treatment of maxima and minima.

1. A man A is 5 miles out at sea in a boat opposite a point B and wishes to reach a point C on the shore at a distance of 8 miles from B. If he can walk 4 miles an hour and row 3 miles an hour, find at what point he should land in order to reach C in the shortest possible time.

Suppose that he lands at a point D at distance x from B.

Then the distance that he rows

$$= AD = \sqrt{25 + x^2} \text{ miles}$$

and the distance that he walks = $DC = 8 - x$ miles.

$$\therefore \text{Time taken} = t = \frac{AD}{3} + \frac{DC}{4}$$

$$= \frac{\sqrt{25 + x^2}}{3} + \frac{(8 - x)}{4} \quad (1)$$

$$\therefore \frac{dt}{dx} = \frac{d\left(\frac{\sqrt{25 + x^2}}{3}\right)}{dx} + \frac{d\left(\frac{8 - x}{4}\right)}{dx}$$

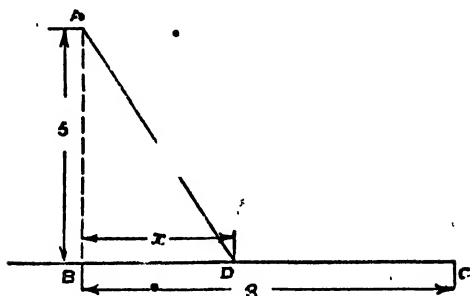


FIG. 18.

$$= \frac{1}{3} \frac{d(25 + x^2)^{\frac{1}{2}}}{dx} + \frac{1}{4} \frac{d(8 - x)}{dx}$$

$$= \frac{1}{3} \frac{(25 + x^2)^{\frac{1}{2} - 1}}{2} \cdot d\left(\frac{25 + x^2}{dx}\right) + \frac{1}{4} (-1)$$

$$= \frac{(25 + x^2)^{-\frac{1}{2}}}{2 \cdot 3} \cdot 2x - \frac{1}{4}$$

$$= \frac{x}{3\sqrt{25 + x^2}} - \frac{1}{4} \quad (2)$$

For a minimum value $\frac{dt}{dx} = 0$

* This is an example of $\frac{dy^n}{dx^n}$, cf. p. 84.

$$\therefore \frac{x}{3\sqrt{25+x^2}} - \frac{1}{4} = 0$$

$$\text{i.e. } 4x = 3\sqrt{25+x^2}$$

$$\text{(squaring) } 16x^2 = 225 + 9x^2$$

$$x^2 = \frac{225}{7}$$

$$x = \frac{15}{\sqrt{7}} = \frac{15\sqrt{7}}{7} = 5.7 \text{ miles, Ans.}$$

(ignoring the negative root).

The corresponding time is 3.102 hours.

We have not yet proved that this is the minimum value but it is easy to do so as follows:—

If $x = 0$, $t = 3.67$ hours

„ $x = 5.7$, $t = 3.102$ „

„ $x = 8$, $t = 3.145$ „

This is the whole range of x and clearly t must be a minimum.

The other and more reliable test is to find $\frac{d^2t}{dx^2}$.

This is not often necessary in practical cases because there is seldom any doubt, but we will do it as an example.

$$\begin{aligned} \frac{d^2t}{dx^2} &= \frac{d}{dx} \left(\frac{x}{3\sqrt{25+x^2}} \right) - \frac{d}{dx} \left(\frac{1}{4} \right) \\ &= \frac{1}{3} \left\{ \frac{\sqrt{25+x^2} \cdot \frac{dx}{dx} - x \frac{d(\sqrt{25+x^2})}{dx}}{(\sqrt{25+x^2})^2} \right\} - 0 \\ &= \frac{1}{3} \left(\frac{\sqrt{25+x^2} - \frac{x^2}{\sqrt{25+x^2}}}{25+x^2} \right) \\ &= \frac{25+x^2 - x^2}{3(25+x^2)^{\frac{3}{2}}} = \frac{25}{3(25+x^2)^{\frac{3}{2}}} \end{aligned}$$

This is positive for $x = 15/\sqrt{7}$ (and in fact for all values of x : note that there is no question as to the sign of the root, as we have adopted and retained the positive determination in the problem). Hence the point is a minimum (§ 36, p. 101).

2. (C) Find the position of a short uniformly distributed load which will give the maximum bending moment at any point of a simply supported girder.

Suppose that the centre of the load, which is of intensity p and length l (Fig. 49), and of total amount $W = pl$, has reached a point D.

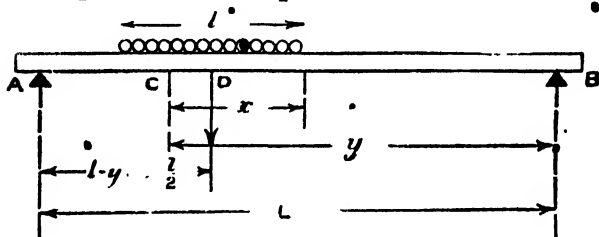


FIG. 49.

Let C be the point at which the bending moment is required, and let the front of the load be at a distance x from C and let the support B be at a distance y from C. Let R_A , R_B be the reactions at A and B, and let $AB = L$.

$$\text{Then } R_B = \frac{W \cdot AD}{L} = \frac{pl}{L} \left(L - y + x - \frac{l}{2} \right)$$

$$\text{Bending moment at C} = M_C = R_B \cdot y - px^2$$

$$= \frac{ply}{L} \left(L - y + x - \frac{l}{2} \right) - \frac{px^2}{2} \quad (1)$$

We want to know for what value of x this bending moment will be a maximum. We therefore treat x

as the variable and put $\frac{dM_C}{dx} = 0$, regarding y as a constant.

$$\begin{aligned}\frac{dM_C}{dx} &= \frac{ply}{L} \frac{d}{dx} \left(L \cdot y + x - \frac{l}{2} \right) - \frac{p}{2} \frac{dx^2}{dx} \\ &= \frac{ply}{L} (0 - 0 + 1 - 0) - \frac{p}{2} \cdot 2x \\ &= \frac{ply}{L} - px \quad (2)\end{aligned}$$

If this = 0, $\frac{ply}{L} = px$

$$\text{or } \frac{x}{l} = \frac{y}{L}$$

We therefore get the rule : *The B.M. at any point is a maximum when the load is in such a position that the given point divides the load in the same ratio as it divides the span.*

If there is any doubt as to whether the result is a maximum or a minimum, and in practical problems there is seldom any doubt, we find $\frac{d^2M_C}{dx^2}$.

$$\text{From (2) } \frac{d^2M_C}{dx^2} = 0 - p.$$

This is negative, therefore the B.M. is a maximum.

3. (M) *The horse-power that can be transmitted by a belt one square inch in section running at V feet per minute is given by the formula*

$$H.P. = \frac{7V}{891} - \frac{V^3}{12,400 \times 10^6} :$$

What speed will give the maximum horse-power and what will be the value of the maximum horse-power?

In this case V is the variable, so we put

$$\frac{d(\text{H.P.})}{dV} = 0.$$

$$\frac{d(\text{H.P.})}{dV} = \frac{7}{891} - \frac{3V^2}{12,400 \times 10^6} = 0$$

$$\therefore V^2 = \frac{12,400 \times 10^6 \times 7}{3 \times 891}$$

$$V = 5700 \text{ feet per minute.}$$

Putting this value for V in the formula for horse-power, we get

$$\text{max. H.P.} = 29.9$$

4. (E) Find the relative values of the internal and external resistance of an electric battery to obtain the greatest power therefrom.

If R_i and R_e are the internal and external resistances respectively, C the current and V the voltage, we have

$$C = \frac{V}{R_i + R_e}.$$

$$\text{The power given out} = W = C^2 R_e$$

$$\therefore W = \frac{V^2 R_e}{(R_i + R_e)^2}$$

regarding R_e as a variable we put $\frac{dW}{dR_e} = 0$.

$$\text{Then } \frac{dW}{dR_e}$$

$$= V^2 \left[(R_i + R_e)^{-2} \cdot \frac{dR_e}{dR_e} - R_e \cdot \frac{d(R_i + R_e)^{-2}}{dR_e} \right] \div (R_i + R_e)^4$$

$$= 0.$$

$$\text{i.e. } (R_i + R_e)^2 - R_e \cdot 2(R_i + R_e) \cdot 1 = 0$$

$$(R_i + R_e)(R_i - R_e) = 0$$

$$\text{i.e. } R_i = R_e$$

on rejecting the solution $R_i + R_e = 0$.

∴ the maximum power is obtained when the external resistance is equal to the internal resistance.

Alternative solution.—In problems of this kind when the denominator is more complicated than the numerator, we can often find the minimum value of the quantity more easily by finding the maximum of the reciprocal and vice versa,

$$\text{e.g. } \frac{1}{W} = \frac{1}{V^2} \left(\frac{R_i^2 + 2R_i R_e + R_e^2}{R_e} \right)$$

$$= \frac{1}{V^2} (R_i^2 R_e^{-1} + 2R_i + R_e)$$

$$\frac{d}{dR_e} \frac{1}{W} = \frac{1}{V^2} \left(-\frac{R_i^2}{R_e^2} + 1 \right) = 0$$

$$\text{i.e. } R_e^2 - R_i^2 = 0 \text{ or } R_e = R_i.$$

To test whether this gives a maximum, differentiate again, thus getting

$$\begin{aligned} \frac{d}{dR_e} \frac{1}{V^2} \left(1 - \frac{R_i^2}{R_e^2} \right) &= \frac{1}{V^2} \{ 0 - R_i^2 (-2)/R_e^{-3} \} \\ &= \frac{2R_i^2}{R_e^3 V^2} \end{aligned}$$

This is positive, therefore the above value is a minimum for $\frac{1}{W}$ or a maximum for W .

5. *The regulations of the parcel post stipulate that a parcel must not exceed six feet in length and girth combined. What is the size of the parcel of the greatest volume that can be sent?*

Let l be the length in feet of the parcel, then the girth or perimeter of the parcel will be $(6 - l)$ feet.

Now if A is the area of the cross section, we have $V = A \cdot l$.

For a given perimeter the circle has a larger area than any other figure,* therefore V will be greatest for a given length for a right cylinder.

$$\text{Now } A = \pi r^2 \\ \text{and } r = \frac{\text{perimeter}}{2\pi} = \frac{6-l}{2\pi}$$

$$\therefore A = \frac{(6-l)^2}{4\pi}$$

$$\therefore V = \frac{l(6-l)^2}{4\pi}$$

$$\frac{dV}{dl} = \frac{1}{4\pi} \frac{d(36l - 12l^2 + l^3)}{dl} = 0$$

$$\text{i.e. } 36 - 24l + 3l^2 = 0$$

$$12 - 8l + l^2 = 0$$

$$(l-6)(l-2) = 0$$

$$\therefore l = 2 \text{ or } 6.$$

6 is clearly the length for minimum volume, since the girth would then be 0.

Therefore the length of the parcel is 2 feet and its diameter $= \frac{4}{\pi} = 1.273$ feet.

The volume will be $\pi r^2 l = \pi \times \left(\frac{2}{\pi}\right)^2 \times 2 = \frac{8}{\pi} = 2.546$

cubic feet.

6. (C) If the velocity of flow through a pipe is proportional to \sqrt{m} for a given fall of head and m is

* The mathematical proof of this depends upon the Calculus of Variations and cannot be given here. The fact will be familiar. For example, if the inner tube of a bicycle tyre is inflated the pressure of the air inside will make the internal volume, and therefore the cross section, as large as possible before stretching the rubber; the cross section becomes circular. A sphere is the surface of given area which contains the greatest volume, as may be inferred from the shape of a soap bubble.

area of water flowing
wetted perimeter of pipe, find how full a circular pipe should run to give a maximum discharge.

Now flow per second = velocity \times area

$$\begin{aligned} q &= k \sqrt{m} \times A \\ &= k \sqrt{\frac{A}{P}} \times A \end{aligned}$$

where P = wetted perimeter = ADB (Fig. 50) and k is a constant coefficient.

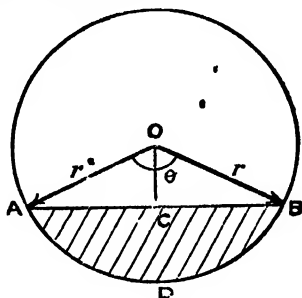


FIG. 50.

Now $P = r\theta$

$$\begin{aligned} A &= \text{area sector OADB} - \text{area } \triangle OAB \\ &= \frac{r^2\theta}{2} - r^2 \sin \theta \end{aligned}$$

(Since the area of a triangle = $\frac{1}{2}$ product of two sides \times sine of contained angle.)

$$= \frac{r^2}{2} (\theta - \sin \theta)$$

$$\therefore q = k \sqrt{\frac{r^2 (\theta - \sin \theta)}{2r\theta}} \cdot \frac{r^2}{2} (\theta - \sin \theta).$$

This will be a maximum when

$$\sqrt{1 - \frac{\sin \theta}{\theta}} \cdot (\theta - \sin \theta)$$

is a maximum,

i.e. when $\theta \left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{3}{2}}$ is a maximum

$$\text{i.e. when } \frac{d}{d\theta} \left\{ \theta \left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{3}{2}} \right\} = 0,$$

$$\text{i.e. } \theta \frac{d}{d\theta} \left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{3}{2}} + \left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{3}{2}} \frac{d\theta}{d\theta} = 0$$

$$\text{i.e. } \theta \cdot \left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{1}{2}} \cdot \frac{d}{d\theta} \left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{3}{2}}$$

$$+ \left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{3}{2}} = 0$$

$$\text{i.e. } \left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{1}{2}} \left\{ \frac{3\theta}{2} \left(-\frac{\theta \cos \theta + \sin \theta}{\theta^2} \right) \right. \\ \left. \left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{3}{2}} \right\} = 0$$

or rejecting the factor $\left(1 - \frac{\sin \theta}{\theta}\right)^{\frac{1}{2}}$

$$\theta - \sin \theta - \frac{3\theta \cos \theta}{2} + \frac{3 \sin \theta}{2} = 0,$$

$$\text{i.e. } \theta = \frac{\sin \theta}{3 \cos \theta}$$

We cannot solve this equation by direct means so we proceed by trial, tabulating as follows (see next page).

It is fairly clear that the pipe must be nearly full to give a maximum discharge, so we will start with $\theta = 300^\circ$. We pass over the choice of trial values necessary to approach the neighbourhood of the correct value, and give a few trials close round it.

* This is an example of the differentiation of a product (cf. p. 82).

| θ° | $\sin \theta$ | $\cos \theta$ | $2 - 3 \cos \theta$ | $\frac{-\sin \theta}{2 - 3 \cos \theta}$ | θ radians |
|----------------|---------------|---------------|---------------------|--|------------------|
| 300 | -.8660 | .5000 | .5000 | 1.74 | 5.238 |
| 305 | -.8192 | .5736 | .2792 | 2.93 | 5.238 |
| 307 | -.7986 | .6018 | .1946 | 4.11 | 5.358 |
| 308 | -.7880 | .6157 | .1529 | 5.14 | 5.877 |
| 310 | .7660 | .6428 | .0716 | 10.69 | 5.410 |

These values can be plotted if the exact result is required, but it is near enough to take $\theta = 308^\circ$.

$$\text{Then depth of water} = r \left(1 + \cos \frac{(360^\circ - 294^\circ)}{2} \right)$$

$$= r (1 + \cos 26^\circ)$$

$$1.8988 r = 919 \times \text{diam.} \quad \text{Ans.}$$

7. (M) The efficiency of a square-threaded screw or a worm gear is given by the formula $\eta = \frac{\tan \alpha}{\tan (\alpha + \phi)}$, where α is the angle of the thread and ϕ is the angle of friction. For a given value of ϕ find what angle of thread will give a maximum efficiency.

$$\eta = \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

The maximum value of η is given by $\frac{d\eta}{d\alpha} = 0$

$$\text{i.e.} \quad \tan (\alpha + \phi) \frac{d \tan \alpha}{d \alpha} - \tan \alpha \frac{d \tan (\alpha + \phi)}{d \alpha} = 0$$

One possible solution of this is $\frac{1}{\tan^2 (\alpha + \phi)} = 0$

$$\text{i.e.} \quad \alpha + \phi = \frac{\pi}{2}$$

but this would make $\eta = 0$ and clearly does not give a maximum. Multiplying through by $\tan^2 (\alpha + \phi)$ we get

$$\tan (a + \phi) \sec^2 a - \tan a \sec^2 (a + \phi) = 0$$

$$\text{i.e. } \frac{\sin^3 (a + \phi)}{\cos (a + \phi) \cos^2 a} - \frac{\sin a}{\cos a \cos^2 (a + \phi)} = 0.$$

Multiplying through by $\cos^2 a \cdot \cos^2 (a + \phi)$,

$$\sin (a + \phi) \cos (a + \phi) \cos^2 a - \sin a \cos a = 0$$

$$\text{i.e. } \frac{1}{2} \sin 2 (a + \phi) - \frac{1}{2} \sin 2 a = 0$$

$$\text{i.e. } \sin 2 (a + \phi) = \sin 2 a$$

$$\text{i.e. } 2 (a + \phi) = 180^\circ - 2 a$$

$$4 a = 180^\circ - 2 \phi$$

$$a = 45^\circ - \frac{\phi}{2}. \quad \text{Ans.}$$

8. (E) *The force exerted by a circular current of radius a on a small magnet whose axis coincides with the axis of the circle varies as $\frac{x}{(a^2 + x^2)^{\frac{5}{2}}}$, where x is the distance of the magnet from the plane of the circle. Prove that the force is a maximum when $x = \frac{1}{2}a$.*

Here $F = \frac{Kx}{(a^2 + x^2)^{\frac{5}{2}}}$, where K is a constant.

$\frac{dF}{dx} = 0$ will give the maximum value of F

$$\therefore \frac{(a^2 + x^2)^{\frac{5}{2}} \frac{dx}{dx} - x \frac{d(a^2 + x^2)^{\frac{5}{2}}}{dx}}{(a^2 + x^2)^5} = 0$$

$$(a^2 + x^2)^{\frac{5}{2}} \cdot 1 - x \cdot \frac{5}{2} \cdot (a^2 + x^2)^{\frac{3}{2}-1} \cdot \frac{dx^2}{dx} = 0$$

$$(a^2 + x^2)^{\frac{5}{2}} - \frac{5x}{2} (a^2 + x^2)^{\frac{3}{2}} \cdot 2x = 0$$

$$(a^2 + x^2)^{\frac{3}{2}} \{ (a^2 + x^2) - 5x^2 \} = 0$$

$$\therefore a^2 - 4x^2 = 0$$

$$\text{or } x = \pm \frac{a}{2}$$

The positive and negative results represent positions on either side of the circle.

§ 38.—**Points of Inflexion in General.**—We have already mentioned points of inflexion in §§ 3, 13, and 36. They are points such as I' (Fig. 28) where the direction of curvature is reversed, i.e. where the curve is momentarily straight. It will be noted that the slope $\frac{dy}{dx}$ diminishes (i.e. the tangent turns in a clockwise direction) as we approach F when it reaches a minimum value and then increases. Hence $\frac{d^2y}{dx^2} = 0$ at a point of inflexion, but $\frac{dy}{dx}$ does not vanish in general: the cases where it does vanish, i.e. when the point of inflexion has a horizontal tangent were mentioned in § 36.

Taking the example already used (p. 100)

$$y = 2r^3 - 3r^2 + 1,$$

we have

$$\frac{d^2y}{dr^2} = 12r - 6 = 0$$

∴ the point of inflexion occurs when

$$r = \frac{1}{2}.$$

EXAMPLE OF POINT OF INFLEXION.—An engineering formula in which the point of inflexion is of interest is Rankine's formula for mild steel columns and struts which may be written

$$f_P = \frac{6}{1 + \frac{c^2}{6000}},$$

where c is the buckling factor (length divided by least radius of gyration for pin ends) (see p. 60, Ex. 2).

Expressing this in general terms we have

$$f_P = \frac{a}{1 + bc^2},$$

where a, b are constants.

$$\frac{df_P}{dc} = \frac{(1 + bc^2)^0 \cdot 0 - a(2bc)}{(1 + bc^2)^2}$$

$$= -\frac{2abc}{(1 + bc^2)^2}$$

$$\frac{d^2f_P}{dc^2} = 0 = -2ab \frac{d}{dc} \frac{c}{(1 + bc^2)^2}$$

$$= -2ab \left[\frac{1 \cdot (1 + bc^2)^2 - c \cdot 2(1 + bc^2) \cdot 2bc}{(1 + bc^2)^4} \right]$$

$$= -2ab \frac{1 + bc^2 - 4bc^2}{(1 + bc^2)^3} = -2ab \frac{1 - 3bc^2}{(1 + bc^2)^3}$$

Equating the numerator to zero,

$$1 - 3bc^2 = 0$$

$$\text{or } c = \pm 1/\sqrt{3b}$$

The positive value is the only possible one

$$\therefore c = \frac{1}{\sqrt{3}} = \sqrt{2000} / 6000$$

$$= 44.7 \text{ approx. Ans.}$$

Compare this with the result obtained by plotting in Exercise II, 2.

EXERCISES 5B.

Further problems (mostly more difficult) on maxima and minima.

1. Find the values of x which will give respectively the maximum and minimum values of

$$2x^3 + 3x^2 + 36x + 8.$$

2. Find the maximum and minimum values of $\frac{(x-1)^2}{(x+1)^3}$.

3. Prove that $\frac{\log x}{x}$ has a maximum value when $x = e$.

4. The efficiency of a Pelton wheel is given by the formula $\eta = \frac{4v}{U^2} (U - v)$, where U is the velocity of the impinging water and v is the velocity of the vane. Find the maximum efficiency and the relative values of U and v at which it occurs.

5. If the strength of a beam of breadth b and depth d varies as bd^2 , show that the strongest beam that can be cut out of a circular log of diameter D has a breadth of .577 D .

6. In steaming against a tide, the resistance of the ship may be taken as proportional to the square of the velocity and the cost of fuel per hour is proportional to the product of the resistance and the velocity. Find the most economical velocity relatively to the tide for going a given distance.

7. In planning a bridge of several spans over a river, the cost of main girders for each span may be taken as proportional to the square of its span. Show that the most economical arrangement of equal spans will arise when the spans are such that the cost of main girders for each span is equal to the cost of each pier.

8. The bending moment at a certain point at distance x from one end of a beam of span l is equal to $\frac{Wx}{3} - \frac{Wx^3}{3l^3}$. Find the maximum bending moment.

9. The flow of steam through an orifice is proportional to $a \sqrt[1]{1 - a^{\frac{\gamma-1}{\gamma}}}$ where a is the ratio of pressures on the two sides and γ is a constant. Find the value of a to give a maximum flow.

10. Divide the number 20 into two parts such that their product is a maximum.

11. Find the least area of canvas that can be used to construct a conical tent of cubic capacity 800 cubic feet.

12. If a triangle has a given base and the sum of the other two sides is given, show that the area is greatest when these two sides are equal.

13. A rectangle of sides x, y has four square corners cut out and the sides turned up so as to form a rectangular box. Show that for the maximum volume the depth will be $\frac{1}{3}\{(x+y) - \sqrt{x^2 + y^2 - xy}\}$.

14. Find the height of a cylinder which is cut out of a sphere of radius R to give the maximum curved surface to the cylinder.

15. The efficiency of a certain hydraulic motor is given by

$$\eta = \frac{v(U - v)}{U^2} - \frac{t}{2} \left(\frac{U - v}{U} \right)^2,$$

where v is the variable. Prove that the maximum efficiency is attained when

$$2v^2U^2 - 4v^3U + gtU^2 + gtv^2 = 0.$$

ANSWERS TO EXERCISES.

5A.

1. $26\frac{1}{2}$ at $x = 2\frac{1}{2}$. Maximum. 2. $14\cdot7, -2\cdot7$.
3. $\pm 6\sqrt{3}$. 4. $21\cdot125$. 6. Half the area of the Δ .
7. Maximum at $x = 1$, inflexion at $x = 2$.
9. $2a^2b^2$. 10. $\frac{\pi}{4}$.

5B.

1. Maximum when $x = -2$ and minimum when $x = +3$. 2. $\frac{2}{27}$; 0. 4. 1; $v = \frac{U}{2}$. 6. $1\frac{1}{2}$ times the velocity of the current. 7. If P = cost of each pier, n = number of spans, l each span and L = total span and cost each span = $4 \cdot \left(\frac{L}{n}\right)^2$, there are

$(n - 1)$ piers, therefore total cost $= (n - 1) P + na l^2$

$\left(\frac{P}{2l}\right) P + Lal$. Then differentiate with respect to l . 8. 128' Wls 9. $\alpha = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}}$

10. 10 and 10. 11. 361 square feet. 13. Let z = height which will be the side of each square corner cut out, therefore dimensions of base of box will be $x - 2z$, $y - 2z$ and volume $z (x - 2z) (y - 2z)$. Differentiate with regard to z and equate to 0 thus getting a quadratic equation with the required solution. 14. 1.114 R.

CHAPTER VI.

SYSTEMATIC INTEGRATION.

§ 39.—**Integrals.**—We have already discussed integrals from the graphical point of view in Chap. I, §§ 6, 7. We will now resume this discussion. Let us suppose y to be given in terms of x ; suppose, for instance,

$$y = x^3.$$

We can then determine the slope of y ,

$$\frac{dy}{dx} = 3x^2.$$

Conversely, if we started with

$$y_1 = 3x^2,$$

then x^3 would be the *indefinite integral** of y , and we should write

$$\dagger \int y dx = \int 3x^2 dx = x^3.$$

Graphically we should say that

$$y = 3x^2$$

being the derived curve of $y = x^3$, it follows that

$$y = 3x^2$$

is the sum curve of $y = 3x^2$.

Integration then is the inverse process to differ-

* We call it "indefinite" because it is the general expression for the integral; in any given problem we have a particular or "definite" integral determined by the special conditions of the problem (see later).

† Read this *integral* $y dx$.

entiation. To *integrate any expression* means to answer the question: what other expression differentiated will give this, one? Although graphically integration is simpler, and more satisfactory than differentiation, analytically it is very much more complex and difficult. The processes of differentiation and integration are related to each other exactly in the same way as multiplication and division. We know that $72 \div 9 = 8$ because we remember that $8 \times 9 = 72$. It will be noted that whenever we differentiate an expression, we learn an integral, e.g. by differentiating x^3 we learn the integral of $3x^2$; this is almost our only method of integrating; and in fact for many expressions, the integral cannot even be expressed at all in terms of the three categories of laws dealt with in Chap. II. This is the case for example with

$$\int \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{and} \quad \int e^{-x^2} dx,$$

both of which arise in some problems.

On the other hand, the integration of a large number of simple expressions is easy enough, and this is so with the greater part of those occurring in a course of engineering.

In the first place we can make a table of standard integrals from our list of standard differentiations in § 32.

LIST OF STANDARD INTEGRALS.

| y | $\int y dx^*$ |
|---------------------------------|--|
| 1. x^n (except $n = -1$) | $\frac{x^{n+1}}{n+1}$ |
| 2. $\frac{1}{x}$ | $\log x$ |
| 3. $\sin x$ | $-\cos x$ |
| 4. $\cos x$ | $\sin x$ |
| 5. $\tan x$ | $\log \sec x$ |
| 6. $\cot x$ | $\log \sin x$ |
| 7. $\sec x$ | $\log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right)$ $= \log \left(\frac{1 + \sin x}{\cos x} \right)$ $= \log \left(\frac{\cos x}{1 - \sin x} \right)$ |
| 8. $\operatorname{cosec} x$ | $\log \tan \frac{x}{2}$ |
| 9. $\sec^2 x$ | $\tan x$ |
| 10. $-\operatorname{cosec}^2 x$ | $\cot x$ |
| 11. $1/\sqrt{1-x^2}$ | $\sin^{-1} x$ |
| 12. $1/(1+x^2)$ | $\tan^{-1} x$ |
| 13. $1/(1-x^2)$ | $\frac{1}{2} \log \frac{1+x}{1-x}$ |
| 14. e^x | e^x |
| 15. $\log x$ | $x \log x - x$ |

Some of these do not occur in the table of differentiations, but all are important, and the student should endeavour to learn them by heart. *All should be verified by differentiation.*

Arbitrary Constant.—The differential coefficient

* $\int dx$ means $\int 1 dx$ and is x .

of a constant is zero. Hence if any arbitrary constant be added to an indefinite integral, it will still do equally well. Hence *to all the indefinite integrals which we obtain an arbitrary constant C must be supposed to be added.* Thus the general form for

$$\int x^n dx \text{ is } \frac{x^{n+1}}{n+1} + C$$

This point is well brought out by a diagram (Fig. 51). The effect

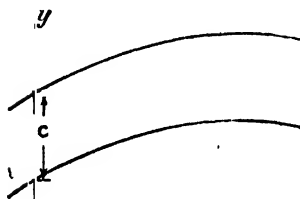


FIG. 51.—Effect of adding a constant to an expression.

of adding a constant C to an expression is to raise the graph through a height C , as in the curves shown. The two curves, however, have everywhere the same slope and therefore the same slope

curve, and the two expressions of which these are the graphs will be integrals of the same expression.

The student can now determine a number of integrals for himself. Thus to integrate $(x^2 + 1)$, we see at once that this will be obtained by differentiating $(\frac{1}{3}x^3 + x + C)$.

EXERCISES 6A.

Evaluate the following indefinite integrals:—

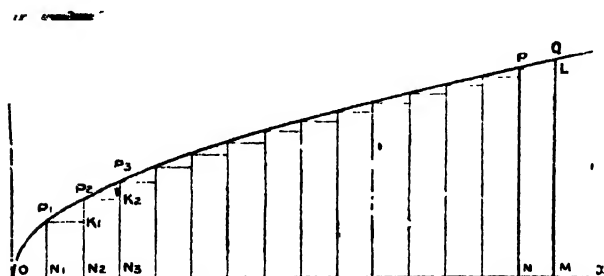
1. $\int (x+1) dx$. 2. $\int (bx+a) dx$. 3. $\int (ax^2+bx+c) dx$.
4. $\int \frac{dx}{\sqrt{x}}$. 5. $\int \frac{3dx}{x^{1.41}}$. 6. $\int \left(\frac{x+2}{x} \right) dx$.
7. $\int (1 - \cos 2x) dx$. 8. $\int \tan^2 x dx = \int (\sec^2 x - 1) dx$.

9. $\int (A \sin mx + B \cos mx) dx$. 10. $\int \frac{dx}{1 + a^2 x^2}$

§ 40. — **Definite Integrals.** — There are two aspects of integration: it first appears as the inverse process to differentiation, and then as a summation. We come now to the second aspect, and we shall afterwards establish the connexion between the two.

Consider the curve (a parabola)

$$y = \sqrt{x} = x^{\frac{1}{2}}$$



$$y = x^{\frac{1}{2}} = \sqrt{x}$$

FIG. 52.

Let NP be any ordinate, where ON = x , and let us try to find the area between the curve OP and the abscissa ON.

Divide the abscissa ON into a large number n of equal parts $ON_1 = N_1N_2 = N_2N_3 = \dots = \frac{x}{n}$. Draw the corresponding ordinates N_1P_1 , N_2P_2 , N_3P_3 , \dots and draw P_1K_1 , P_2K_2 , \dots parallel to the x axis.

Then the sum of the rectangles N_1K_1 , N_2K_2 , \dots is less than the area under the curve but approaches it in value, the defect being the sum of the

triangular pieces $ON_1P_1, P_1K_1P_2, P_2K_2P_3, \dots$. If the number n of slices into which we cut the area is increased the approximation will become closer, for the triangular pieces form a vanishing area bordering the curve.

So that when n is sufficiently great, the error is below our limits of accuracy, and the sum of the rectangles becomes the area under the curve.

The quantities $\frac{x}{n}$ may be considered as increments in x and written δx , while $0, N_1P_1, N_2P_2, \dots$ being successive ordinates may be written $y_0 = 0, y_1, y_2, \dots$

Thus the area under the curve is ultimately equal to

$$y_1\delta x + y_2\delta x + y_3\delta x + \dots$$

Using the symbol S to mean "*the sum of all expressions like,*" the area

$$= \sum_{x=0} S y \delta x,$$

the extreme values of the abscissa being indicated below and above the S . When n is sufficiently great for the limits of accuracy to be surpassed (i.e. the error of the approximation is less than the errors of calculation, measurement, etc.), we lengthen the S :

$$A = \text{Area} = \int_0^x y dx,$$

"the integral of y from 0 to x ".

We can suppose that we take the area for different values of the variable x , and then A would have an expression in terms of x . Let us consider what expression this is:

Let us consider a small change in A

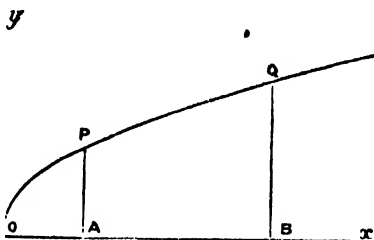
$$\begin{aligned}
 \delta A &= \text{area NMQP} \\
 &\approx \text{rectangle NMLP (approx.)} \\
 &= NP \times NM \\
 &= y \delta x.
 \end{aligned}$$

So that

$$\begin{aligned}
 y &= \frac{\delta A}{\delta x} \text{ (approx.)} \\
 &= \frac{dA}{dx} \text{ (exactly).}
 \end{aligned}$$

So that A is an expression which when differentiated gives y (or $x^{\frac{1}{2}}$), i.e. A is an integral of $y = x^{\frac{1}{2}}$.

Referring to our table of integrals (No. 1, p. 123) we see that



$$y = x^{\frac{1}{2}} = \sqrt{x}.$$

FIG. 53.

$$\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3} x^{\frac{3}{2}} + C.$$

But we do not yet know what value of C must be taken. To find what value C has in this particular case, note that $A = 0$ when $x = 0$, and substitute

$$0 = \frac{2}{3} 0^{\frac{3}{2}} + C, \text{ i.e. } C = 0.$$

Thus

$$A = \frac{2}{3} x^{\frac{3}{2}} = \frac{2}{3} xy \text{ i.e. } \frac{2}{3} \times \text{rectangle ON} \times \text{NP}$$

Suppose we wished to find the area of a different section say between $x = a$ and $x = b$

Then area up to $A = \frac{2}{3} a^3 + C$ (retaining C for the moment) and area up to $B = \frac{2}{3} b^3 + C$. Hence

$$\text{area ABQP} = \int_a^b y dx = \left(\frac{2}{3} b^3 + C \right) - \left(\frac{2}{3} a^3 + C \right) \\ = \frac{2}{3} b^3 - \frac{2}{3} a^3.$$

So that C disappears and any indefinite integral could have been taken. This is why we have not put in the arbitrary constants in the list of standard integrals.

These integrals with upper and lower limits are called *definite integrals*, and to find a definite integral, first find the indefinite integral, then substitute in it the upper and lower limits, and then subtract the latter result from the former.

In practice the definite integral expresses a result under definite specified conditions, while the indefinite integral expresses the same result when the conditions are not specified. The former, for example, might be the distance covered by a moving body starting from a given point, the latter the distance covered from any starting-point.

The following are examples: note the square bracket notation:—

$$x^n dx = \frac{x^{n+1}}{n+1} = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

$$\int_a^b e^x dx = [e^x]_a^b = e^b - e^a$$

$$\int_0^{\frac{\pi}{2}} \sin x dx = \left[-\cos x \right]_0^{\frac{\pi}{2}} = \left(-\cos \frac{\pi}{2} \right) - (-\cos 0) \\ = 0 - (-1) = 1,$$

$$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 \\ = \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4} = .785.$$

Note.—We have supposed that the divisions δx were all equal: the value of the integral is not altered if they are not all equal, provided only that they all become indefinitely small. The proof of this is too difficult to insert here.

EXERCISES 6B.

Evaluate the following definite integrals:—

1. $\int_1^4 x^2 dx.$
2. $\int_0^1 dx.$
3. $\int_{10}^{20} 2(4+x) dx$
4. $\int_{-1}^1 (x-x^3) dx.$
5. $\int_1^4 \frac{dx}{2\sqrt{x}}.$
6. $\int_{-1.5}^6 (2x+3)^2 dx.$
7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$
8. $\int_0^1 \frac{dx}{1+x^2}.$
9. $\int_{\frac{\pi}{2}}^{\pi} \sin^2 x dx.$
10. $\int_0^{\pi} \sec^2 x dx.$

§ 41. Applications of Integrals.—Applications of both indefinite and definite integrals are very varied and numerous. We shall discuss several in some detail in the next chapter. But we will give a few instances here.

Consider again the nature of an integral. We take the values of a dependent quantity y for a whole sequence of very near values of x , we multiply each by the corresponding increment and we add. Any quantity z which is obtained from another y in this way is an integral, and the area of a curve is only a special case.

1°. MASS OF A ROD OF VARYING DENSITY.—Let the length of the rod be 1 ft., and let its density ρ , i.e. its mass per unit length, at any point be proportional to the distance from one end, and be 3 lb. per foot at the other end.

The density at distance x from the front end = $\frac{1}{3} \times 3 = 3x$, so that $\rho = 3x$.

Suppose the rod to be divided into a number of equal parts of which δx is one.

Then the mass of this part = $\rho \delta x$ (approx.), and the whole mass is

$$\begin{aligned} & \sum \rho \delta x \quad (\text{approx.}) \\ & \int_0^1 \rho dx \quad (\text{exactly}). \end{aligned}$$

$$\begin{aligned} \text{So mass} &= \int_0^1 \rho dx = \int_0^1 3x dx \\ &= 3 \times \frac{1}{2} = \frac{3}{2} \text{ lb.} \end{aligned}$$

2'. TO FIND THE VELOCITY AT ANY INSTANT AND THE DISTANCE FALLEN THROUGH BY A BODY FALLING FREELY UNDER GRAVITY.—The body is dropped from a given point, and moves with a uniform acceleration = 32 feet per second per second.

If x is the distance it has fallen in time t , then the velocity is $\frac{dx}{dt}$ and the acceleration

$$\frac{d^2x}{dt^2} = 32.$$

Now $\frac{dx}{dt}$ is the quantity which, when differentiated, gives 32. So

$$\begin{aligned} \frac{dx}{dt} &= \int_0^t 32 dt = [32t]_0^t \\ &= 32t. \end{aligned}$$

Similarly x is the integral of the velocity

$$\begin{aligned}
 x &= \int_0^1 32t \, dt \\
 &= \left[32 \cdot \frac{t^2}{2} \right]_0^1 = 32 \cdot \frac{1^2}{2} \\
 &= 16 t^2.
 \end{aligned}$$

§ 42. **Reduction to Standard Form.**—So far we know the fifteen integrals given in the standard list of § 39, and such others as we may arrive at by differentiation. We can obtain the other integrals by reducing them to standard ones by various means.

1°. **MULTIPLYING BY A CONSTANT.**—Looking at an integral from the summation point of view we see that the integral of *a times y* = *a times the integral of y*, i.e. $\int a y dx = a \int y dx$.

For example

$$\int a \sin x \, dx = -a \cos x.$$

2°. **MULTIPLYING THE ARGUMENT * BY A CONSTANT.**—We will find the integral of $\sin bx$,

$$\int \sin bx \, dx.$$

It is clear that an increment in bx

$$\delta bx = b \delta x.$$

If then in the sum we replace each increment δx by δbx , we shall be multiplying each element and therefore the whole by b . Hence

$$\begin{aligned}
 \int \sin bx \, dx &= \frac{1}{b} \int \sin bx \, d(bx) \\
 &= \frac{1}{b} (-\cos bx) \\
 &= -\frac{1}{b} \cos bx.
 \end{aligned}$$

* The *argument* of $\sin x$ is x , of $\sin bx$ is bx , and in general of $f(x)$ is x .

3°. ADDITION OF CONSTANT TO ARGUMENT.—Consider the integral of $\sin (x + c)$. An increment in $(x + c)$ is the same as an increment in x , viz. δx ,

$$\text{i.e. } \delta (x + c) = \delta x.$$

Hence

$$\begin{aligned} \int \sin (x + c) dx &= \int \sin (x + c) d (x + c) \\ &= -\cos (x + c). \end{aligned}$$

Combining 1°, 2° and 3°

$$\begin{aligned} \int a \sin (bx + c) dx &= a \cdot \frac{1}{b} \int \sin (bx + c) d (bx + c) \\ &= -\frac{a}{b} \cos (bx + c); \end{aligned}$$

To take another example

$$\begin{aligned} \int a^x dx &= \int (e^{\log_e a})^x dx \\ &= \int e^{x \log_e a} dx \\ &= \frac{1}{\log_e a} \int e^{x \log_e a} d (x \log_e a) \\ &= \frac{1}{\log_e a} e^{x \log_e a} \\ &= \frac{a^x}{\log_e a}. \end{aligned}$$

And another

$$\begin{aligned} \int \log_{10} x dx &= \int (\log_{10} e \times \log_e x) dx \\ &= \log_{10} e \int \log_e x dx \\ &\quad \text{because } \log_{10} e \text{ is a constant} \\ &= \log_{10} e (x \log_e x - x) \\ &= x \log_{10} e \log_e x - x \log_{10} e \\ &= x \log_e x - x \log_{10} e. \end{aligned}$$

4°. THE INTEGRAL OF A SUM is the sum of the corresponding integrals. Thus

$$\begin{aligned} \int (x^2 + 3x + 2) dx &= \int x^2 dx + 3 \int x dx + 2 \int 1 dx \\ &= \frac{x^3}{3} + 3 \frac{x^2}{2} + 2x. \end{aligned}$$

5°. TRANSFORMATION OF "INTEGRAND".—Consider the following cases:—

$$\begin{aligned}
 (1) \int (x+1)^2 dx &= \int (x^2 + 2x + 1) dx \\
 &= \int x^2 dx + 2 \int x dx + \int dx \\
 &= \frac{x^3}{3} + 2 \cdot \frac{x^2}{2} + x \\
 &= \frac{1}{3} (x^3 + 3x^2 + 3x).
 \end{aligned}$$

This may also be done as follows by 3°:—

$$\begin{aligned}
 \int (x+1)^2 dx &= \int (x+1) d(x+1) \\
 &= \frac{1}{3} (x+1)^3 \\
 &= \frac{1}{3} (x^3 + 3x^2 + 3x + 1).
 \end{aligned}$$

Observe that these two integrals of the same expression differ by a constant, viz. $\frac{1}{3}$. This does not matter as they are *indefinite*.

$$\begin{aligned}
 (2) \int \sin^2 x dx &= \int \frac{1}{2} (1 - \cos 2x) dx \\
 &= \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x dx \\
 &= \frac{1}{2} \int dx - \frac{1}{4} \int \cos 2x d(2x) \\
 &= \frac{1}{2} x - \frac{1}{4} \sin 2x \\
 &= \frac{1}{2} (x - \cos x \sin x).
 \end{aligned}$$

6°. PARTIAL FRACTIONS. — To integrate $\frac{1}{1-x^2}$,

we may write

$$\frac{1}{1-x^2} = \frac{1}{2} \cdot \frac{1}{1-x} + \frac{1}{2} \cdot \frac{1}{1+x} \quad (1)$$

Then

$$\begin{aligned}
 \int \frac{dx}{1-x^2} &= \int \left[\frac{1}{2} \cdot \frac{1}{1-x} + \frac{1}{2} \cdot \frac{1}{1+x} \right] dx \\
 &= \frac{1}{2} \int \frac{dx}{1-x} + \frac{1}{2} \int \frac{dx}{1+x} \\
 &= -\frac{1}{2} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1} \\
 &= -\frac{1}{2} \log(x-1) + \frac{1}{2} \log(x+1) = \log \sqrt{\frac{x+1}{x-1}}
 \end{aligned}$$

To effect the transformation (1), write

$$\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

where A and B are "undetermined coefficients".

Multiply up by $1-x^2$, then

$$1 = A(1+x) + B(1-x) \quad (2)$$

Putting $x = 1$, $1 = 2A$, and $A = \frac{1}{2}$.

Putting $x = -1$, $1 = 2B$, and $B = \frac{1}{2}$.

Or we may "equate the coefficients" on the two sides. Thus (2) is

$$1 = A + B + (A - B)x.$$

$$\text{Hence } 1 = A + B, 0 = A - B,$$

whence, as before, $A = B = \frac{1}{2}$.

§ 43.—Transformation of Independent Variable.

—This is the most useful method for reducing to standard form, and depends upon the "function of a function" formula. If y depends upon z and z depends upon x , we have (§ 27, 4°)

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Suppose we have an expression in z which we wish to integrate, say,

$$y_1 = \frac{dy}{dz}.$$

Then our integral is

$$y = \int y_1 dz.$$

But we may not be able to integrate this and so find y . We may then use our formula which gives us the slope of y with respect to another variable x ; for we have

$$y = \int \frac{dy}{dx} dx = \int \frac{dy}{dz} \cdot \frac{dz}{dx} dx,$$

that is to say an integration in terms of x instead of z ; and this new integral we may be able to find.

Suppose we wish to integrate $\frac{1}{\sqrt{1-z^2}}$ (one of our standard integrals).

We want then

$$\int \frac{dz}{\sqrt{1-z^2}}$$

Put $z = \sin x$. The integral

$$\begin{aligned} \int \frac{dz}{\sqrt{1-z^2}} &= \int \frac{1}{\sqrt{1-\sin^2 x}} \frac{dz}{dx} dx \\ &= \int \frac{1}{\sqrt{1-\sin^2 x}} \cos x dx \\ &= \int \frac{1}{1-\sin^2 x} \cos x dx \\ &= \int \frac{\cos x}{\cos^2 x} dx = \int dx = x. \end{aligned}$$

(Note that when integrating with respect to x we must put everything in terms of x .)

But we want our integral in terms of z : we have then

$$\text{integral} = x = \sin^{-1} z.$$

The general formula may be written

$$\int y_1 dz = \int y_1 \frac{dz}{dx} dx,$$

and therefore it comes to replacing

$$dz \text{ by } \frac{dz}{dx} dx.$$

This is in reality only the formula for a small variation, $\delta z = \frac{dz}{dx} \delta x$.

It will be asked how we arrive at the particular transformation

$$z = \sin x.$$

How do we know this will enable us to perform the integration? We do not know. We can only make trials of likely transformations until we arrive at the right one. But a little experience will soon put us upon the right track. It may be remarked that a radical like

$$\sqrt{1 - z^2}$$

may be reduced by putting $z = \sin x$, and similarly for

$$\sqrt{z^2 + 1} \text{ we put } z = \tan x$$

$$\sqrt{z^2 - 1} \text{ we put } z = \sec x.$$

The formula may often be applied from the opposite point of view, i.e. we can regard our original integral as having the form

$$\int y_1 \frac{d}{dx} dx;$$

and put it into the form

$$\int y_1 dz.$$

Examples:—

$$(1) \quad \int_x^1 \log x \, dx:$$

note that $\frac{1}{x} = \frac{d \log x}{dx}.$

So we have

$$\begin{aligned} \int \log x \frac{1}{x} dx &= \int \log x \frac{d}{dx} \log x dx \\ &= \int \log x \, d \log x = \frac{1}{2} (\log x)^2 \end{aligned}$$

(by standard integral No. 1).

$$(2) \quad \int x \sin x^2 dx = \frac{1}{2} \int \sin x^2 : 2x dx$$

$$(2x dx = dx^2)$$

$$= \frac{1}{2} \int \sin x^2 dx^2$$

$$= -\frac{1}{2} \cos x^2$$

(by standard integral No. 3).

$$(3) \int \tan x dx = \int \frac{\sin x dx}{\cos x}$$

$$(- \sin x dx = d \cos x)$$

$$= - \int \frac{d \cos x}{\cos x}$$

$$= - \log \cos x$$

$$= \log \frac{1}{\cos x} = \log \sec x$$

(by standard integral No. 2: this is No. 5).

§ 44. **Integration by Parts.**—This method depends upon the product formula in differentiation (§ 27, 2°).

$$\frac{dy_1 y_2}{dx} = y_2 \frac{dy_1}{dx} + y_1 \frac{dy_2}{dx}$$

Rearranging

$$y_1 \frac{dy_2}{dx} = \frac{dy_1 y_2}{dx} - y_2 \frac{dy_1}{dx}$$

Now integrate both sides of this equation.

$$\int y_1 \frac{dy_2}{dx} dx = y_1 y_2 - \int y_2 \frac{dy_1}{dx} dx$$

[which may also be written (see last §)

$$[y_1 dy_2 = y_1 y_2 - y_2 dy_1].$$

To take an example consider

$$\int x \cos x dx;$$

$\cos x$ is the differential coefficient of $\sin x$. So we have

$$\begin{aligned} \int x \cos x dx &= \int x \frac{d \sin x}{dx} dx \\ &= x \sin x - \int \sin x \frac{dx}{dx} dx \\ &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x. \end{aligned}$$

The original integral is put in terms of another

which can be integrated. This method is useful when we have x (or x^2 , x^3 , etc.) multiplying an integrable expression. The formula may be applied several times. Thus to integrate

$$\begin{aligned}\int x^2 \cos x \, dx &= \int x^2 d \sin x \\ &= x^2 \sin x - \int \sin x \, d x^2 \\ &= x^2 \sin x - \int \sin x \cdot 2x \, dx \\ &= x^2 \sin x - 2 \int x \sin x \, dx \\ &= x^2 \sin x - 2(-x \cos x + \sin x) \\ &= x^2 \sin x + 2x \cos x - 2 \sin x.\end{aligned}$$

Another example is

$$\int \log x \, dx :$$

here we take

$$y_1 = \log x, \quad y_2 = x$$

$$\text{then } \int \log x \, dx = x \log x - \int x \cdot d \log x$$

$$= x \log x - \int x \cdot \frac{1}{x} \, dx$$

$$= x \log x - \int dx$$

$$= x \log x - x.$$

EXERCISES 6C.

Evaluate the following integrals, hints being given in many cases as to the method to adopt:—

1. $\int \frac{dx}{2x-4}$.
2. $\int \frac{dx}{\sqrt{2x+5}}$.
3. $\int \frac{x}{x^2+1} dx$, substituting $z = x^2 + 1$.
4. $\int \frac{dx}{x^2 - a^2}$, working by partial fractions.
5. $\int \frac{5x-1}{(x+1)(2x-1)} dx$, working by partial fractions.
6. $\int \frac{dx}{(x^2+r^2)^{\frac{3}{2}}}$, putting $x = r \tan \theta$.
7. $\int_0^b \frac{x dx}{r^2 + x^2}$.
8. $\int \frac{dx}{2-x-x^2}$ by partial fractions.

- tions. 9. $\int \frac{(x+1)dx}{(x-1)^2}$. 10. $\int \frac{dx}{x(1+x^2)}$ putting $z = x^2$.
 11. $\int \frac{x dx}{x^4 - 1}$ putting $z = x^2$. 12. $\int \frac{x dx}{\sqrt{x^4 - x^2}}$.
 13. $\int \sin^2 x dx$. 14. $\int \cos^2 (nx + \frac{1}{2})$. 15. $\int \sin^2 x \cos x dx$.
 16. $\int_0^\pi \cos^2 x dx$. 17. $\int x \sin x dx$, by parts.
 18. $\int x \sin x \cos x dx$, by parts. 19. $\int x e^x dx$, by parts.
 20. $\int x \sec^2 x dx$. 21. $\int \frac{dx}{(1+x^2)^2}$ putting $x = \tan \theta$.
 22. $\int \frac{x dx}{\sqrt{x+1}}$. 23. $\int \frac{x^2 dx}{1-x^4}$, by partial
 fractions. 24. $\int_0^1 \frac{x dx}{\sqrt{1-x^2}}$. 25. $\int_0^1 \frac{x dx}{\sqrt{1+x^2}}$.
 26. $\int_1^2 \frac{dx}{\sqrt{x^2-1}}$. 27. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\cos \theta}$. 28. $\int \sqrt{\frac{1-x}{1+x}} dx$,
 by putting $x = \cos \theta$ and simplifying.
 29. $\int \frac{dx}{1-4x+4x^2}$ putting $z = (2x-1)$ 30. $\int \tan^2 x dx$.

ANSWERS TO EXERCISES.

EXERCISES 6A.

1. $\frac{x^2}{2} + x$. 2. $\frac{bx^2}{2} + ax$. 3. $\frac{ax^3}{3} + \frac{bx^2}{2} + cx$.
 4. $2\sqrt{x}$. 5. $\frac{-3}{41x^{41}}$. 6. $x + 2 \log x$. 7. $x - \frac{\sin 2x}{2}$.
 8. $\tan x - x$. 9. $\frac{B}{m} \sin mx - \frac{A}{m} \cos mx$.
 10. $\frac{1}{a} \tan^{-1} ax$.

EXERCISES 6B.

1. 21. 2. 1. 3. 380. 4. 0. 5. 1. 6. 553.5.
 7. $\frac{\pi}{2}$. 8. $\frac{\pi}{4}$. 9. $\frac{\pi}{4}$. 10. infinity.

EXERCISES, 60.

1. $\frac{1}{2} \log (2x - 4)$. 2. $\sqrt{(2x + 5)}$
3. $\frac{1}{2} \log (x^2 + 1)$. 4. $\frac{1}{2a} \log \left(\frac{x - a}{x + a} \right)$. 5. $4 \log (x + 1)$.
6. $-\frac{3}{2} \log (2x - 1)$. 7. $\frac{1}{2} \log \frac{r^2 + b^2}{r^2}$.
8. $\frac{1}{3} \log \frac{x + 2}{1 - x}$. 9. $\log (x - 1) - \frac{2}{x - 1}$.
10. $\log \frac{x}{\sqrt{1 + x^2}}$. 11. $\frac{1}{4} \log \frac{x^2 - 1}{x^2 + 1}$. 12. $\frac{1}{2} \sin^{-1} \frac{x^2}{r^2}$.
13. $\frac{1}{2} x - \frac{1}{4} \sin 2x$. 14. $\frac{x}{2} + \frac{1}{4n} \sin 2(nx + b)$.
15. $\frac{\sin^3 x}{3}$. 16. $\frac{\pi}{2}$. 17. $\sin x - x \cos x$.
18. $\frac{\sin 2x}{8} - \frac{x \cos 2x}{4}$. 19. $a(x - a)e^{\frac{x}{a}}$.
20. $x \tan x + \log \cos x$. 21. $\frac{1}{2} \tan^{-1} x + \frac{1}{2} \left(1 + \frac{x}{x^2} \right)$.
22. $\frac{2}{3} (x - 2) \sqrt{x + 1}$. 23. $\frac{1}{2} \log \frac{1 + x}{1 - x} - \frac{1}{2} \tan^{-1} x$.
24. 1. 25. $\sqrt{2} - 1$. 26. $\log (2 + \sqrt{3})$. 27. 1.
28. $\sin^{-1} x + \sqrt{1 - x^2}$. 29. $-\frac{1}{2(2x - 1)}$.
30. $\tan x - x$.

CHAPTER VII.

APPLICATIONS AND DEVELOPMENTS OF INTEGRATION.

§ 45.—We will now consider some of the simpler problems in engineering work in which integration arises. We wish here again to emphasize the fact that all integration is a matter of practice and experience, and in some problems we may get expres-

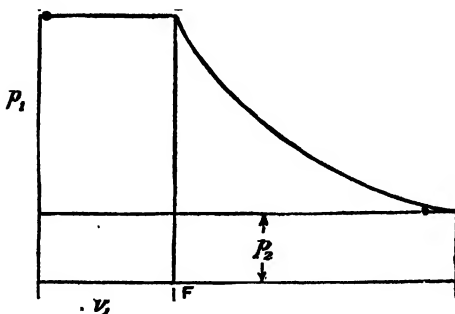


FIG. 54.

sions to integrate that we cannot work out without very great trouble. When such an integration occurs we can always get a result which will be approximately correct by plotting against the variable the expression that we wish to integrate, and finding the

area of the resulting figure by planimeter or otherwise (Chapter I).

§ 46. **Work done in Engine Cylinder.**—Suppose that we have a steam or other heat engine in which gas or vapour is admitted at constant pressure p_1 up to a point B where it is cut off (Fig. 54). It then expands to a point C, and is then exhausted at constant pressure p_2 .

Taking unit area of piston, the pressure represents the force on the piston and the volume represents the position in the stroke from the beginning. We have shown already (p. 19) that the work done is equal to the area of the diagram of force plotted against distance.

$$\begin{aligned}\therefore \text{work done} &= \text{area ABCD} \\ &= \text{area (ABFO + FBCE - ODCE)} \\ &= p_1 v_1 + \int_{v_1}^{v_2} p dv - p_2 v_2.\end{aligned}$$

Now in most gases the expansion can be expressed by the law $p v^n = \text{constant} = k$.

$$\begin{aligned}\therefore \int p dv &= \int \frac{k dv}{v^n} = k \int \frac{dv}{v^n} \\ &= k \int v^{-n} dv = \frac{k v^{-n+1}}{-n+1} \\ &= \frac{k v}{(1-n) v^n} = \frac{p v}{(1-n)} \\ \therefore \int p dv &= \frac{p_2 v_2 - p_1 v_1}{(1-n)} = \frac{p_1 v_1 - p_2 v_2}{(n-1)}\end{aligned}$$

ADIABATIC EXPANSION.—In this case n , in the equation $p v^n = k$ is γ , the ratio between the specific heats at constant pressure and volume respectively.

$$\begin{aligned} \therefore \int_{v_1}^{v_2} p dv &= \frac{p_1 v_1 - p_2 v_2}{(\gamma - 1)} \\ \therefore \text{work done} &= \text{area ABCD} \\ &= p_1 v_1 - p_2 v_2 + \frac{p_1 v_1^\gamma - p_2 v_2^\gamma}{\gamma - 1} \\ &= (p_1 v_1 - p_2 v_2) \left(1 + \frac{1}{\gamma - 1} \right) \\ &= \frac{\gamma}{\gamma - 1} (p_1 v_1 - p_2 v_2) \end{aligned}$$

ISOTHERMAL EXPANSION.—In this case n is 1, i.e. $p v = k$, $\therefore p_1 v_1 = p_2 v_2$.

Our expression $\frac{p_1 v_1 - p_2 v_2}{(n - 1)}$ then becomes $\frac{0}{0}$ which is *indeterminate*.

We must therefore go back to the integral which gives

$$\begin{aligned} \int p dv &= k \int \frac{dv}{v} = k \log_e v \\ \therefore \int_{v_1}^{v_2} p dv &= k (\log_e v_2 - \log_e v_1) = k \log_e \frac{v_2}{v_1} \end{aligned}$$

The quantity $\frac{v_2}{v_1}$ is commonly called r , the ratio of expansion.

$$\begin{aligned} \therefore \int_{v_1}^{v_2} p dv &= k \log_e r = p_1 v_1 \log_e r \\ \therefore \text{work done} &= \text{area ABCD} \\ &= p_1 v_1 + p_1 v_1 \log_e r - p_2 v_2 \\ &= p_1 v_1 \log_e r \text{ (since } p_1 v_1 = p_2 v_2 \text{)}. \end{aligned}$$

§ 47. **Mean Values.**—Let the diagram represent a varying quantity y . Construct a rectangle $OBTS$ having the same area as that under the curve. Then

* Because the logarithm of a quotient is $\log \text{dividend} - \log \text{divisor}$.

OS, the height of the rectangle, is the mean value of y between the limits O and B .

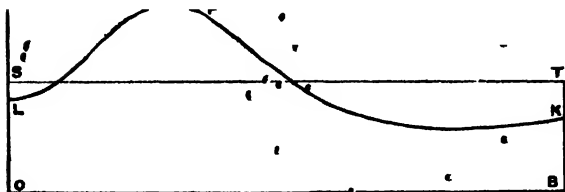


FIG. 55.

Let $OB = a$; then the area $OBKPL = \int_0^a y dx$.

Hence

$$\text{the mean value of } y = \frac{1}{a} \int_0^a y dx.$$

If \bar{y} = the mean value, then $ay = \int_0^a y dx$.

Examples —

1°. The mean value of $\sin x$ between $x = 0$ and $x = \pi$

$$\begin{aligned} \int_0^\pi \sin x \, dx &= \left[-\cos x \right]_0^\pi \\ &= \frac{1}{\pi} \left(-(-1) - (-1) \right) \\ &= \frac{2}{\pi} \end{aligned}$$

2°. In the last article (§ 46) the mean pressure of the fluid between A and C will be $\frac{n(p_1 v_1 - p_2 v_2)}{v_2(n-1)}$ and

between B and C will be $\frac{P_1 v_1 - P_2 v_2}{(r_2 - r_1)(n - 1)}$.

§ 49 Centres of Gravity.—FIRST MOMENTS.—

Taking two co-ordinate axes OX, OY in a plane, suppose that particles of matter m_1, m_2, m, \dots are situated at the points P_1, P_2, P_3, \dots

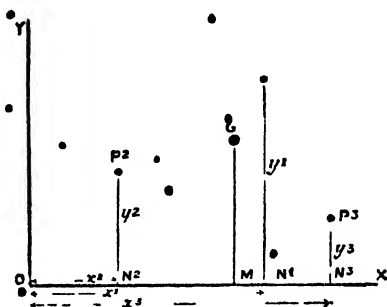


FIG. 56

Then the first moment of the particle m_1 about the axis OX is

$$m_1 y_1,$$

and the first moment of the system in m_1, m_2, m_3, \dots is the sum of the first moments of the particles taken separately,

first moment of system about OX

$$= m_1 y_1 + m_2 y_2 + m_3 y_3 + \dots$$

Similarly for the first moment about any other line, for example about OY, we have

$$m_1 x_1 + m_2 x_2 + m_3 x_3 + \dots$$

CENTRE OF GRAVITY.—Suppose now that the whole mass

$$M = m_1 + m_2 + m_3 + \dots$$

is concentrated at a point G, such that the first

moments of M at G about OX and about OY are equal respectively to the first moments of the system about OX and about OY . If the co-ordinates of G are (\bar{x}, \bar{y}) , then

$$\bar{M}\bar{x} = m_1x_1 + m_2x_2 + m_3x_3 + \dots \text{ (about } OY \text{)}$$

$$\text{and } \bar{M}\bar{y} = m_1y_1 + m_2y_2 + m_3y_3 + \dots \text{ (about } OX \text{)}.$$

Then it can be proved that the first moment of M at G and the first moment of the system, about any other axis are also equal. So that as far as first

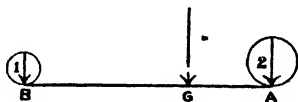


FIG. 57.

moments are concerned, the whole mass may be supposed concentrated at its centre of gravity.

In particular the moment of the system about any axis through G is zero.

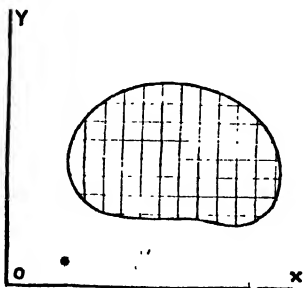


FIG. 58.

[Example: the centre of gravity of a mass of 2 lb. at A and a mass of 1 lb. at B is on AB and divides it so that $2AG = GB$.]

§ 49. **Continuous Bodies.**—Continuous bodies may be supposed to be divided up into a large number of small parts as in Fig. 58, each of which may be treated as a particle. If we call each part δm (an "element of mass"), then the co-ordinates of the centre of gravity may be determined as before by the equations

$$M\bar{x} = x_1\delta m + x_2\delta m + x_3\delta m + \dots$$

$$My = y_1\delta m + y_2\delta m + y_3\delta m + \dots$$

approximately, or by

$$M\bar{x} = \int x dm, \quad My = \int y dm.$$

In some elementary cases the position of G may be inferred from symmetry. Thus for uniform plates

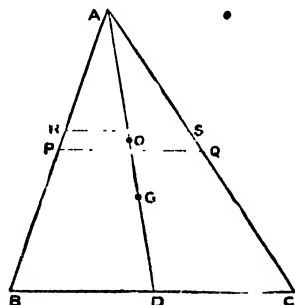


FIG. 59.

in the shapes of a square, a parallelogram, or a circle, it is clear that the centre of the figure is also the centre of gravity.

Again take a uniform triangular plate ABC; divide it up into thin rods such as PQSR parallel to the base BC. Then the median AD bisects each rod, and hence passes through the centre of gravity of each rod. Hence the triangle may be replaced by a

number of particles placed along AD, and G must lie in AD. Similarly G must lie in each of the two other medians and must therefore be their point of intersection. And hence G is a point on DA, one-third of the way along it.

CENTROIDS.—In dealing with areas we often want to find the *centroid*, i.e. the point in the area which would be the centre of gravity of a uniform plate of the same shape as the area. In this case we proceed exactly as before, replacing mass by area in the expressions. In this case the formulæ become

$$Ax = \int x da, \quad Ay = \int y da$$

where A = total area of body
 da = element of area.

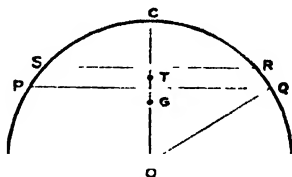


FIG. 60.

§ 50. **Examples.**—We will now take a few cases of centres of gravity and centroids in which integration is useful.

1°. **A UNIFORM SEMICIRCULAR PLATE.**—In the figure (Fig. 60) it is clear that G must lie upon the symmetrical radius OC.

Let $OG = y$; and suppose we cut the semicircle into rods such as PQRS, parallel to the base AB. Let $y = OT$ and let the radius be a . Then the thickness of a rod is δy , and its length

$$TQ = \sqrt{(OQ^2 - OT^2)} = \sqrt{a^2 - y^2}.$$

Its mass = $2\rho \sqrt{a^2 - y^2} \cdot \delta y$, where ρ = the mass of unit area. The first moment about AB

$$= 2\rho \sqrt{a^2 - y^2} \delta y \times y.$$

Hence

$$\begin{aligned} M\bar{y} &= 2 \int_0^a \rho \sqrt{a^2 - y^2} y dy \\ &= \rho \int_0^a \sqrt{a^2 - y^2} \cdot 2y dy \\ &= \rho \int_0^a \sqrt{a^2 - y} \frac{dy'}{dy} \cdot dy * \\ &= \rho \int_0^a \sqrt{a^2 - y} dy \\ &= \rho \int_0^a (a - y) d(a - y') \\ &= \rho \left[(a - y)^2 \right]_0^a \\ &= \rho (0 - a^2) \\ &= \rho a^2. \end{aligned}$$

But $M = \rho \cdot \frac{1}{2} \pi a^2$.

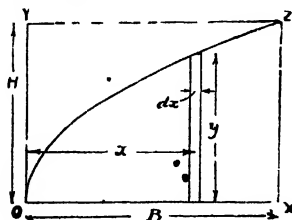


FIG. 61.

Hence

$$OG = \bar{y} = \frac{\rho a^3}{\frac{1}{2} \pi \rho a^2} = \frac{4a}{3\pi}.$$

2. A PARABOLA.—Consider next the parabola

* This is an example of the method explained in § 43.

$y^2 = 4ax$, and take the area between the curve and the axis of x .

$$\text{Area of curve} = \int y dx = \int_a^B 2a^{\frac{1}{2}} x^{\frac{1}{2}} dx$$

$$2a^{\frac{1}{2}} \int_0^B x^{\frac{1}{2}} dx = a^{\frac{1}{2}} \cdot \frac{2}{3} x^{\frac{3}{2}}$$

$$= \frac{4}{3} a^{\frac{1}{2}} B^{\frac{3}{2}}.$$

Now $2a^{\frac{1}{2}} B^{\frac{3}{2}} = \Pi$.

$$\therefore \text{Area of curve} = \frac{2}{3} BH.$$

$$\text{First moment about OY} = \int xy dx = \int_a^B 2a^{\frac{1}{2}} x^{\frac{3}{2}} dx$$

$$2a^{\frac{1}{2}} \int_0^B x^{\frac{3}{2}} dx = 2a^{\frac{1}{2}} \cdot \frac{2}{5} x^{\frac{5}{2}} \Big|_0^B$$

$$= \frac{4}{5} a^{\frac{1}{2}} B^{\frac{5}{2}} = \frac{2}{5} B^2 H.$$

$$\therefore \text{distance of centroid from OY} = \frac{\frac{2}{5} B^2 H}{\frac{2}{3} BH} = \frac{3}{5} B.$$

To find the distance of the centroid from OX, note that the centroid of each strip is at its middle point.

Hence the first moment about OX

$$\begin{aligned} &= \int_{x=0}^{x=B} \frac{y}{2} \cdot y dx = \frac{1}{2} \int_a^B 4a x dx \\ &= 2a \int_a^B x dx = 2a \cdot \frac{1}{2} B^2 = aB^2. \end{aligned}$$

\therefore distance of centroid from OX

$$= \frac{aB^2}{\frac{2}{3} BH} = \frac{3aB}{2H} = \frac{3}{2H} \cdot \frac{H^2}{4} = \frac{3H}{8}.$$

SURFACES OF REVOLUTION.—The following rules with regard to surfaces of revolution (i.e. surfaces generated by revolving a plane curve about an axis),

commonly known as the "Theorems of Pappus," follow directly from the consideration of centroids.

1. *The volume of a surface of revolution is equal to the area of the surface multiplied by the length of the path of its centroid.*

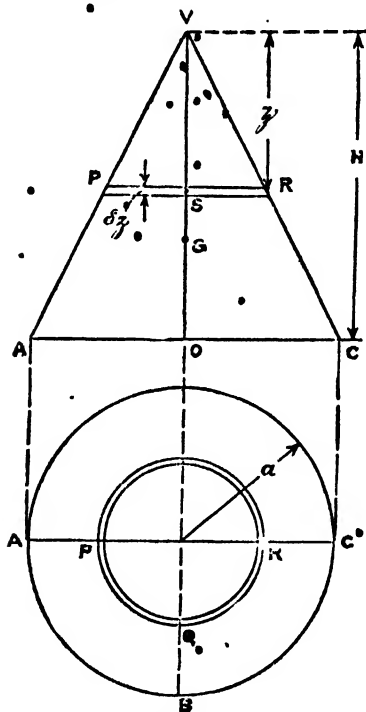


FIG. 62.

2. *The surface of a surface of revolution is equal to the length of the generating arc multiplied by the length of the path of its centroid.*

3°. A UNIFORM RIGHT CIRCULAR CONE.—In our

general discussion, we considered only flat plates. The extension to solid bodies is easy, but here we must consider first moments not about axes but *about planes*.

Let OV be the axis of the cone, V being the vertex: and let any distance measured along the axis VS = z .

Cut the cone into slices such as PSR of thickness δz , parallel to the base ABC. Then the centre of gravity (S) of each plate lies on the axis VO, and hence the centre of gravity G of the cone lies on VO.

Suppose then the mass of each plate to be concentrated at its centre of gravity, and take moments about a plane through V parallel to the base.

If a = radius of base,

h = height OV,

ρ = mass per unit volume (density),

then

$$\text{the radius of a plate} = SR = OC \frac{VS}{VO} = a \frac{z}{h};$$

$$\text{the area of the plate} = \pi \left(a \frac{z}{h} \right)^2 = \pi \frac{a^2}{h^2} z^2$$

$$\therefore \text{its mass} = \pi \frac{a^2}{h^2} z^2 \delta z \rho.$$

\therefore The first moment

$$\begin{aligned} & \int_0^h \pi \frac{a^2}{h^2} z^2 \cdot z \cdot \rho \delta z \\ &= \pi \rho \frac{a^2}{h^2} \int_0^h z^3 dz = \pi \rho \frac{a^2}{h^2} \cdot \frac{h^4}{4} = \frac{\pi \rho}{4} a^2 h^2. \end{aligned}$$

And the mass of the cone

$$\begin{aligned} &= \frac{1}{3} \text{area of base} \times \text{height} \times \text{density} \\ &= \frac{1}{3} \pi a^2 \cdot h \cdot \rho = \frac{\pi \rho}{3} a^2 h. \end{aligned}$$

Hence

$$\frac{\pi p}{3} a^2 h \cdot VG = \frac{\pi p}{4} a^2 h^2$$

Or

$$VG = \frac{3}{4} h = \frac{3}{4} VO.$$

§ 51. **Moment of Inertia and Radius of Gyration.**—**DEFINITION.**—The *moment of inertia* or *second*

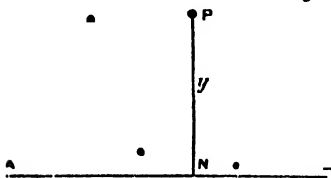


FIG. 63.

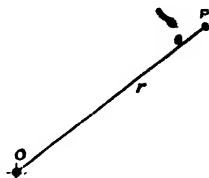


FIG. 64.

moment of a particle m situated at P about an axis or line AB is

$$mPN^2 = my^2,$$

where y is the perpendicular distance of P from AB .

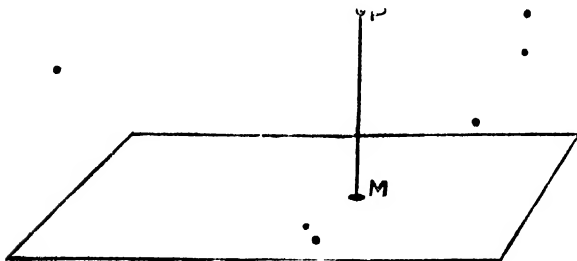


FIG. 65.

The *second moment* of a particle m at P about a point O or axis through O perpendicular to the plane of the paper is

$$m \cdot OP^2 = mr^2.$$

This is often called a *polar second moment*.

The second moment of a particle m at P about a plane is similarly

$$mPM^2 = mz^2,$$

where z is the perpendicular distance of P from the plane.

The second moment of a system of particles is the sum of the second moments of the particles taken separately. This applies equally to a continuous solid body which can be supposed sub-divided into a large number of small parts each of which is to be treated as a particle.

The *polar moment of inertia of a wheel*, of radius a and mass M (the mass of the spokes and axle being neglected) is clearly Ma^2 , every point of the rim being at the same distance from the axis.

MOMENT OF INERTIA OF AN AREA.—Although an area has no mass and can therefore strictly have no inertia, the second moment of an area, i.e. the sum of the products of each element of area by the square of its distance from a line or axis, is of great importance in a number of problems upon beams and shafts and is usually called the *moment of inertia of the area*. All the properties that we prove for masses apply equally well to areas.

RADIUS OF GYRATION.—In the same way that in considering first moments of bodies or areas we can regard the whole area as concentrated at a point called the centre of gravity c centroid, we can also in dealing with moments of inertia regard the whole area as concentrated at a point which we call the *secondroid* which will depend upon the axis taken. Then the distance of this point from the axis or line about which moments have been taken is called the

radius of gyration about the axis and is given the letter k . The value of k can be found only by the relation

$$Mk^2 \text{ or } Ak^2 = I$$

$$\text{i.e. } k = \sqrt{\frac{I}{M}} \text{ or } \sqrt{\frac{I}{A}}$$

The first form is used for solid bodies and the second for areas.

The radius of gyration is a very important quantity and comes into a very large number of problems in constructional design as well as in problems dealing with rotating and vibrating bodies.

§ 52. Two Properties of Second Moments. --
1°. (a) Consider the second moments of a system:—

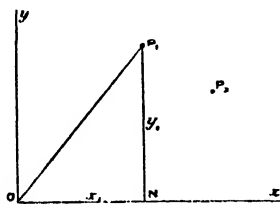


FIG. 66.

(1) about an axis OX,

$$I_x = m_1 y_1^2 + m_2 y_2^2 + \dots$$

$$= \Sigma m y^2$$

(where Σ means "the sum of terms like").

(2) about a perpendicular axis OY

$$I_y = m_1 x_1^2 + m_2 x_2^2 + \dots$$

$$= \Sigma m x^2$$

(3) about their point of intersection O,

$$J = m_1 r_1^2 + m_2 r_2^2 + \dots$$

$$= \Sigma m r^2$$

A, B, C = second moments about the planes
YOZ, ZOY, XOY (respectively).

Then clearly

$$I_X = B + C, I_Y = C + A, I_Z = A + B$$

It can also be proved that

$$I_P = A + B + C = \frac{1}{2} (I_X + I_Y + I_Z)$$

$$\therefore k_P^2 = \frac{1}{2} (k_X^2 + k_Y^2 + k_Z^2).$$

2°. PARALLEL AXES.—If

I = the moment of inertia of a plate about any axis
AB in its plane,

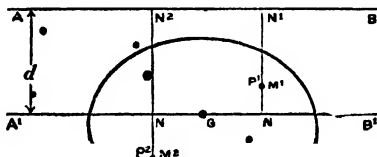


FIG. 68.

I_c = the moment of inertia about a parallel axis
A'B' through its centre of gravity G,

d = perpendicular distance of G from AB,

M = mass of plate,

then

$$I = I_c + Md^2.$$

For (see Fig. 68) and let $P^1 N^1 = y_1$ and $P^2 N^2 = y_2$;
 $P^1 N = y_1'$ and $P^2 N = y_2'$

$$\begin{aligned} I &= m_1 y_1^2 + m_2 y_2^2 + \dots = \sum m y^2 \\ &= m_1 (y_1' + d)^2 + m_2 (y_2' + d)^2 + \dots = \sum m (y_1' + d)^2 \\ &= \sum m (y_1'^2 + 2d y_1' + d^2) \\ &= \sum m y_1'^2 + 2 \sum m d y_1' + \sum m d^2 \\ &= \sum m y_1'^2 + 2d \sum m y_1' + d^2 \sum m \\ &= I_c + 2d \cdot 0 + d^2 \cdot M \\ &= I_c + Md^2 \end{aligned}$$

* This is 0 because the sum is the first moment of the system
about an axis through the centre of gravity which is 0 (§ 48,
p. 146).

Note.—The perpendicular is considered positive when drawn *upwards*, so that (in the figure) P_2N is positive, while P_1N is negative, and always

$$y_1 = y_1' + d.$$

Exactly the same formula expresses the moment of inertia of a solid body about any axis in terms of a parallel axis through the centre of gravity.

Similarly for areas we have

$$I = I_c + Ad^2.$$

Corresponding to this we have for solids

$$Mk^2 = Mk_c^2 + Md^2$$

and for areas

$$Ak^2 = Ak_c^2 + Ad^2$$

Both these give

$$k^2 = k_c^2 + d^2.$$

It will thus be seen clearly that the radius of gyration is a *length* and the formula comes the same for solid bodies as for areas.

The dimensions of moments of inertia.—Many students find considerable difficulty in getting clear ideas upon moments of inertia; this is, we think, largely because they cannot form a concrete idea of what dimensions the quantity is. In the case of areas the moment of inertia is of the order area \times (length)² = (length)⁴ and it is not easy to get a concrete idea of this dimension; in the case of solids it is of the order mass \times (length)² and this again is not a familiar dimension although it is much easier to conceive. When difficulty of this kind is experienced it is best to regard the moment of inertia as some new idea and not to attempt to form a concrete idea of it as being of the dimension (length).⁴

§ 53. Some Important Moments of Inertia.

1°. UNIFORM ROD.—*Axis through the centre perpendicular to the rod.*—Let the length be l , and the mass M . Subdivide the rod into small lengths $PQ = \delta x$, x being OP .

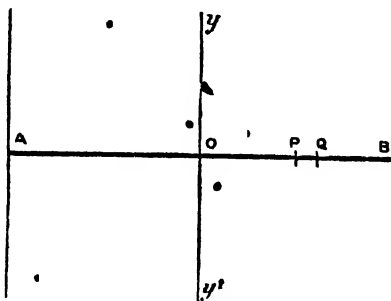


FIG. 69.

The mass per unit length

$$\rho = \frac{M}{l}$$

Hence the mass of $PQ = \frac{M}{l} dx$.

$$I = \text{Moment of inertia} = 2 \int_0^l x^2 \cdot \frac{M}{l} dx.$$

$$= \frac{2M}{l} \int_0^l x^2 dx$$

$$= \frac{2M}{l} \cdot \frac{1}{3} \left(\frac{l}{2} \right)^3 = \frac{Ml^2}{12}.$$

[The integral gives the moment of inertia of one half, OB , of the rod: this is why it is doubled.]

Moment of inertia about axis through A perpendicular to the rod.

We make use of 2° in the last §.

$$I = I_c + Md^2 = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}.$$

2°. UNIFORM RECTANGULAR PLATE.—Let the mass = M and the sides be of length B and H .

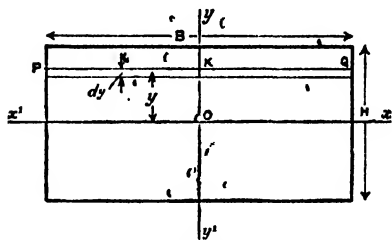


FIG. 70.

Divide the rectangle into rods such as PQ of thickness dy , parallel to $X'X$, y being OK .

Then the mass of a rod = $\frac{M}{H} \cdot dy$.

And M. I. (moment of inertia) about $X'X$ = I_x

$$= 2 \int_0^H \frac{M}{H} \cdot y^2 dy = \frac{2M}{H} \cdot \left(\frac{H}{2}\right)^3 \cdot \frac{1}{3}$$

$$= \frac{MH^2}{12}.$$

In fact this is really the same case as that of the rod.

The M. I. about $Y'Y$ = $I_y = \frac{MB^2}{12}$. And by § 52, 1°

the polar M. I. about axis through O perpendicular to the rectangle

= sum of M. I.'s about XX' and $Y'Y$

$$= \frac{M}{12} (B^2 + H^2).$$

Similarly for the rectangular area we have, since the area $A = BH$ corresponds to M ,

$$I_x = \frac{BH^3}{12}$$

$$I_y = \frac{HB^3}{12}$$

$$J = \frac{A}{12} (B^3 + H^3).$$

3°. THIN WHEEL (spokes and axle neglected).

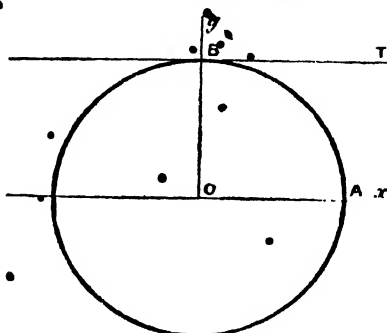


Fig. 71.

Let M = mass.

• R = radius.

Then J (polar moment of inertia) about the axis is

$$J = MR^2.$$

This is the same clearly as the M. I. about its centre O .

To find the M. I. about a *diameter*.

We have (§ 52, 1°)

$$J = I_x + I_y$$

(OX and OY being perpendicular)

$$= 2 \text{ (M. I. about OX)}$$

$$\therefore \text{M. I. about OX} = \frac{1}{2} MR^2.$$

Using § 52, 2°,

M. I. about a *tangent* BT

$$= \frac{1}{2} MR^2 + MR^2 = \frac{3}{2} MR^2.$$

4°. **UNIFORM CIRCULAR PLATE.**—Let the radius = R .

To find the M. I. about an axis through the centre O perpendicular to the plate, divide the plate up into thin circular strips as shown in the figure.

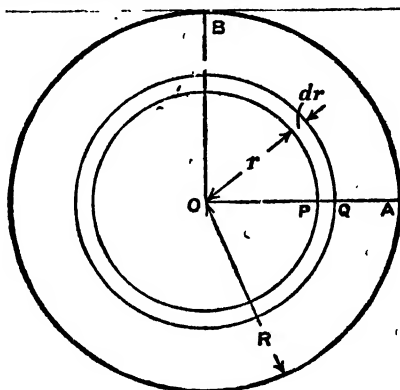


FIG. 72.

If r = radius OP of any strip, and dr = its thickness PQ ,

Then its mass = density \times length \times thickness

$$= \frac{M}{\pi R^2} \times 2\pi r \times dr$$

$$= \frac{2M}{R^2} r dr.$$

$$dJ = \frac{2M}{R^2} r dr \times r^2 = \frac{2M}{R^2} r^3 dr.$$

$$\therefore \text{M. I. of plate} = \int_0^R \frac{2M}{R^2} r^3 dr$$

$$= J = \frac{2M}{R^2} \int_0^R r^3 dr$$

$$= \frac{2M}{R^2} \cdot \frac{R^4}{4} = \frac{1}{2} MR^2.$$

To find the M. I. about a diameter Fig. 72, we have as in 3° (by § 52, 1°)

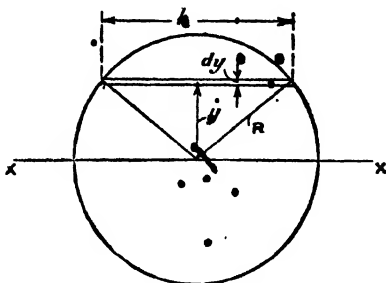


Fig. 73.

$$\frac{1}{2} MR^2 = 2 \text{ (M. I. about diameter)}$$

$$\therefore \text{M. I. about diameter} = \frac{1}{4} MR^2.$$

And again

$$\text{M. I. about tangent (by § 52, 2°)}$$

$$= \frac{1}{4} MR^2 + MR^2 = \frac{5}{4} MR^2.$$

For a circular area we shall have about the diameter

$$I = \frac{\pi R^4}{4}$$

$$= \frac{\pi D^4}{64}$$

where D is the diameter.

The value of the moment of inertia of a circle about its diameter may also be obtained at much greater length as follows; the student should complete it as an exercise.

Referring to Fig. 73

$$I_{xx} = 2 \int_0^R b \cdot dy \cdot y^2.$$

Npw let θ be the angle between the normal y and the radius R .

$$\begin{aligned} \text{Then} \quad b &= 2 R \sin \theta \\ y &= R \cos \theta \\ \therefore \frac{dy}{d\theta} &= -R \sin \theta \end{aligned}$$

$$\begin{aligned} \therefore \int_0^b dy \cdot y^2 &= \int_{\frac{\pi}{2}}^0 R^2 \cos^2 \theta \cdot 2 R \sin \theta (-R \sin \theta) d\theta \\ &= 2 R^4 \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta \end{aligned}$$

Calling the integral B

$$\begin{aligned} \text{we have } B &= \int_0^{\frac{\pi}{2}} (2 \sin \theta \cos \theta)^2 d\theta \\ &= \frac{1}{4} \int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta \\ &= \frac{1}{8} \int_0^{\pi} (1 - \cos 4\theta) d\theta \\ &= \frac{1}{8} \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{16} \end{aligned}$$

$$\text{whence } I = 2 \cdot 2 R^4 \cdot \frac{\pi}{16}$$

$$= \frac{\pi R^4}{4}$$

as before.

The reader will find that he can avoid a great deal of integration by the judicious use of the propositions given in § 52 which are often applicable to symmetrical bodies.

5°. SPHERE.—A similar method applies to a uniform sphere which is subdivided into spherical shells. We find

$$\text{M. I. about diameter} = \frac{2}{5} MR^2$$

$$\text{M. I. about tangent line} = \frac{7}{5} MR^2$$

6°. UNIFORM CYLINDER.—Let the length be $2L$, and the radius be R . To find M. I. about axes.

Take three perpendicular axes:—

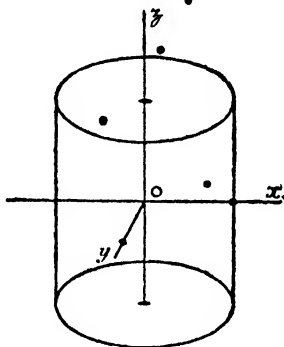


FIG. 74.

Oz , the axis of the cylinder, Ox , Oy , two axes at right angles through the centre perpendicular to the axis, Fig. 74.

Then using the notation of (§ 52, 1°)

$$A = B = \frac{1}{4} MR^2 \text{ by 1°}$$

for this comes to taking the M. I. of a circular plate about a diameter]

$$\text{and } C = \frac{1}{3} ML^2 \text{ by 1° similarly,}$$

we have at once

$$\text{M. I. about axis } OZ = I_z = \frac{1}{2} MR^2$$

$$\text{and M. I. about } OX = I_x = B + C$$

$$= \frac{1}{4} MR^2 + \frac{1}{3} ML^2$$

$$= M \left(\frac{1}{4} R^2 + \frac{1}{3} L^2 \right).$$

7°. PARABOLIC SEGMENT.—Referring to Fig. 61,

Second moment about OY = $\int x^2 y dx = \int 2a^{\frac{1}{2}} x^{\frac{3}{2}} dx$

$$= 2a^{\frac{1}{2}} \int_0^B x^{\frac{3}{2}} dx = \left[2a^{\frac{1}{2}} \cdot \frac{2}{7} x^{\frac{7}{2}} \right]_0^B$$

$$= \frac{4}{7} a^{\frac{1}{2}} B^{\frac{7}{2}} = \frac{2}{7} B^3 H$$

$$\therefore k_Y^2 = \frac{\frac{2}{7} B^3 H}{\frac{2}{3} BH}$$

$$\text{or } k_Y = \sqrt{\frac{3}{7}} B.$$

If the second moment is required about the base XX, we proceed as follows:—

$$I_{OY} = \frac{2}{7} B^3 H.$$

$$I_{CC} = I_{OY} - A \cdot d^2$$

$$= \frac{2}{7} HB^3 - \frac{2}{3} \cdot BH \cdot \frac{9B^2}{25}$$

$$I_{XZ} = I_{CC} + Ad_1^2$$

$$= \frac{2}{7} HB^3 - \frac{6}{25} HB^3 + \frac{8}{75} HB^3$$

$$= 11B^3 \left\{ \frac{150}{5 \cdot 5} - \frac{126}{5 \cdot 5} + \frac{56}{5 \cdot 5} \right\} = HB^3 \cdot \frac{80}{525}$$

$$= \frac{16}{105} HB^3.$$

§ 54. Product Moments.—If dm is an element of mass at distances x, y from two perpendicular axes the quantity $\int xy dm$ is called the *product moment*,

which in the case of areas becomes $\int xy da$. This

quantity bears some resemblance to the moment of inertia and is useful in some more advanced problems in connexion with beams whose sections have no axis of symmetry.

It follows at once that the product moment about an axis of symmetry is zero because corresponding to every point on one side of such axis there is another point on the other side whose ordinate is equal to it numerically but opposite in sign.

§ 55. Centre of Pressure and Load Point (C). —In the case of bodies subjected to fluid pressure such as dams, sluices, dock-gates, etc., and those

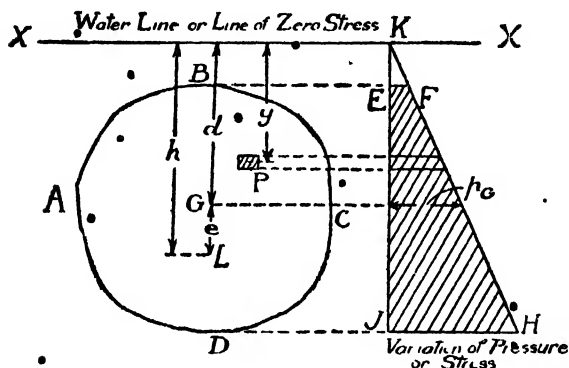


FIG. 75.

subjected to a uniformly varying stress such as beam sections, we require to find the *centre of pressure*, i.e. the point L at which a single force equal to the total pressure or stress upon the body may act to keep it in equilibrium.

In the case of stresses L is often called the *load point*. The depth of the centre of pressure is usually measured from the line XX of zero pressure or stress which is the *water line* in the case of fluid pressure and the *neutral axis* in the case of beams.

Referring to Fig. 75, if da is an element of area

situated around the point P, since the intensity of pressure or stress is proportional to the depth below XX, the intensity of pressure at P = $p_p = \rho \cdot y$ where ρ is some constant, which in the case of fluid pressure is equal to the weight of unit volume of the fluid and in the case of stresses is the intensity of stress at unit distance from the neutral axis.

$$\begin{aligned} \text{The force on element of area} &= dQ = p_p \cdot da \\ &= \rho y da. \end{aligned}$$

$$\begin{aligned} \therefore \text{total force on body} &= Q = \int \rho y da \\ &= \rho \int y da \end{aligned}$$

$$= \rho \times \text{first moment about XX}$$

$$= \rho \cdot A \cdot \bar{d} = A \rho \bar{d} \quad (1)$$

$$= \text{area} \times \text{intensity of pressure or stress at centroid.}$$

Now take moments about the line XX.

$$\begin{aligned} \text{Moment of force on element} &= p_p \cdot y da \\ &= \rho y^2 da. \end{aligned}$$

$$\therefore \text{total moment about XX} = \int \rho y^2 da = \rho I_x$$

$$\therefore \text{if } h \text{ is the depth of the centre of pressure}$$

$$Q \cdot h = \rho I_x$$

$$\text{i.e. } h = \frac{\rho I_x}{Q} = \frac{\rho I_x}{\rho \cdot A \bar{d}}$$

$$= \frac{I_x}{A \bar{d}} \quad (2)$$

$$= \frac{\text{second moment of area about XX}}{\text{first moment of area about XX}}$$

In some cases it is more convenient to find the distance e between the centroid and the centre of pressure. We then use the relation (see p. 157)

$$I_x = I_c + A \bar{d}^2$$

Equation (2) then becomes

$$\begin{aligned} h &= d + e = \frac{I_c}{Ad} + \frac{Ad^2}{Ad} \\ &= \frac{I_c}{Ad} + d \\ \therefore e &= \frac{I_c}{Ad} = \frac{Ak_c^2}{Ad} \\ &= \frac{k_c^2}{d} \end{aligned} \quad (3)$$

In some problems on stress distribution we are given the position of the load point and require to find the distance d from the centroid to the neutral axis: we then use the relation

$$d = \frac{k_c^2}{e} \quad (4)$$

In such problems e is called the *eccentricity* of the load.

Considering the application to stresses we notice that as e , the eccentricity, diminishes, d gets longer, i.e. the neutral axis gets farther away and as a result of this the difference between the maximum and minimum stresses JH and EF will diminish, the stresses ultimately becoming uniform across the section as L coincides with G . The neutral axis can then be regarded as at an infinite distance away; this will be seen mathematically by making e very small in equation (4), because k_c^2 is a finite quantity and a finite quantity when divided by a very small quantity gives a very great quantity.

§ 56. Special Cases of Centre of Pressure.—

1. *Rectangular Surface Extending up to the Water Line.*—Referring to p. 170 if H is the depth below the water line and B the breadth

$$I_x = \frac{BH^3}{3}, A = BH \text{ and } d = \frac{H}{2}.$$

$$\therefore k = \frac{I_x}{Ad} = \frac{1}{3} BH^3 / \frac{1}{2} BH^2 = \frac{2H}{3}$$

Therefore in dock gates, dams, etc., the centre of pressure acts at $\frac{2}{3}$ depth from the water line

Similarly in reinforced concrete rectangular beams

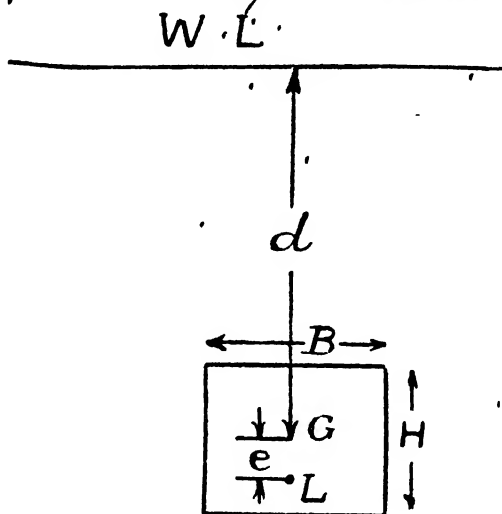


FIG 76

the resulting compression stress is at $\frac{2}{3}$ distance from the neutral axis to the compression edge

2. *Rectangular Surface, Entirely below the Water Line* — k^2 for a rectangle = $\frac{H^2}{12}$ [see p. 160].

$$\therefore \text{in this case } e = \frac{H^2}{12d}.$$

It will be noted that e diminishes as d increases

or that the centre of pressure or load point gets nearer to the centroid of the section as the depth of the upper portion of the area below the water line increases. In the limiting case in which $d = \frac{H}{2}$, $e = \frac{H}{6}$ i.e. $h = \frac{H}{2} + \frac{H}{6} = \frac{2H}{3}$, thus agreeing with the previous case as, of course, it should.

3. *Circular Surface Entirely Below the Water*

Line.—I for a circle about its diameter $= \frac{\pi D^3}{64}$ [p. 163].

$$\therefore k_c^2 = \frac{D^2}{16}$$

$$\therefore \text{in this case } e = \frac{D^2}{16d}$$

This will be less than for a square surface of side equal to D .

§ 57. (M) **Friction on Footstep Bearings.**—*Flat Bearing.*—We will now consider the application of integration to calculate the power absorbed in footstep bearings, Fig 77. In the case of the flat bearing we will assume that the pressure is uniformly distributed over the whole bearing

$$\therefore \text{intensity of pressure} = p = \frac{\text{load}}{\text{area}} = \frac{W}{\pi R^2}$$

\therefore pressure on an elementary ring of width dr situated at a distance r from the centre

$$= dW = p \times \text{area of ring} = p \times 2\pi r dr$$

$$\therefore \text{friction on ring} = \mu dW = \mu p \cdot 2\pi r dr$$

$$\therefore \text{moment of friction on ring about centre}$$

$$= dM_f = \mu dW \cdot r = 2\pi \mu p r^2 dr$$

\therefore moment of friction on whole surface about centre

$$\begin{aligned}
 = M, &= \int_0^R 2\pi \mu p r^2 dr \\
 &= 2\pi \mu p \int_0^R r^2 dr \\
 &= 2\pi \mu p \left[\frac{r^3}{3} \right]_0^R \\
 &= \frac{2\pi \mu p R^3}{3} = \frac{2}{3} \mu p R^3
 \end{aligned}$$

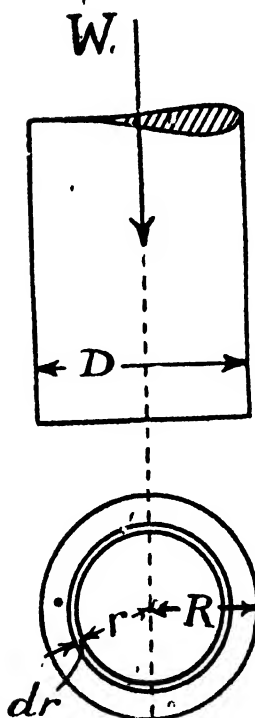


FIG. 77.

∴ work lost in friction per second if n is the number of revolutions per second $= 2\pi n M_f$ (by the rule that work per second $= 2\pi n \times$ moment of force) and R is in inches

$$= \frac{4\pi\mu WRn}{3} \text{ inch-lb.} \quad (1)$$

$$\begin{aligned} \therefore \text{Horse-power absorbed} &= \frac{4\pi\mu WRn}{3 \times 550 \times 12} \\ &= \frac{\mu WRn}{1575} \quad (2) \end{aligned}$$

Collar Bearing.—In this case referring to Fig. 78 we get by exactly similar reasoning

$$\begin{aligned} M_f &= 2\pi\mu p \left| \frac{r}{3} \right. \\ &= \frac{2\pi\mu p}{3} (R_1^3 - R_2^3) \end{aligned}$$

The area in this case $= \pi (R_1^2 - R_2^2)$

$$\begin{aligned} \therefore p &= \frac{W}{\pi (R_1^2 - R_2^2)} \\ \therefore M_f &= \frac{2\pi\mu W}{3} \frac{(R_1^3 - R_2^3)}{\pi (R_1^2 - R_2^2)} \\ &= \frac{2\mu W}{3} \frac{(R_1 - R_2) (R_1^2 + R_1 R_2 + R_2^2)}{(R_1 - R_2) (R_1 + R_2)} \\ &= \frac{2\mu W}{3} \frac{(R_1^2 + R_1 R_2 + R_2^2)}{(R_1 + R_2)} \quad (3) \end{aligned}$$

$$\begin{aligned} \therefore \text{Horse-power absorbed} &= \frac{\mu W (R_1^2 + R_1 R_2 + R_2^2) n}{1575 (R_1 + R_2)} \quad (4) \end{aligned}$$

In the case where the collar is very small and R_1 and R_2 become practically equal, each say equal to R ,

$$M_f = \frac{2\mu W}{3} \cdot \frac{3R^2}{2R} = \mu WR.$$

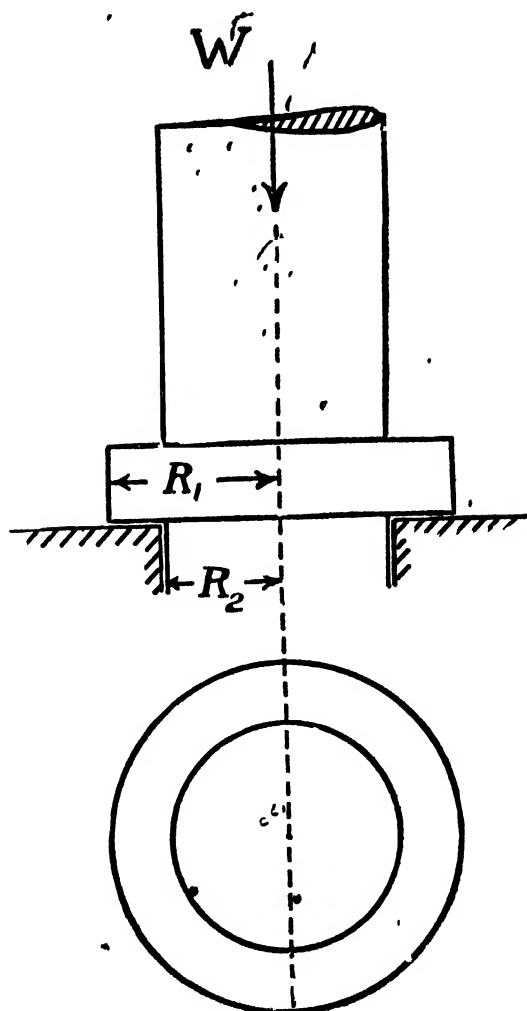


FIG. 78.

Conical Bearing.—Referring to Fig. 79, the normal pressure p if assumed constant will be such that

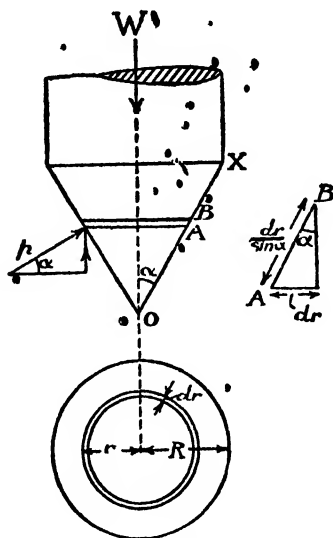


FIG. 79.

$p \times \text{area of cone} = \frac{W}{\sin \alpha}$ and area of cone = OX
 $\times \frac{1}{2}$ circumference of base

$$\therefore p \times \pi R \times OX = \frac{W}{\sin \alpha}$$

$$\text{i.e. } p \times \pi R \times \frac{R}{\sin \alpha} = \frac{W}{\sin \alpha}$$

$$\therefore p = \frac{W}{\pi R^2}$$

This is independent of the angle of the cone and is the same as for the flat pivot.

∴ pressure on ring AB = $p \times 2\pi r$ AB.

$$= \frac{p \times 2\pi r dr}{\sin \alpha}$$

∴ friction on AB acting at right angles to the paper

$$= \mu p \times \frac{2\pi r dr}{\sin \alpha}$$

∴ moment about centre line

$$= dM_f = \frac{\mu p \times 2\pi r dr \cdot r}{\sin \alpha}$$

∴ total moment of friction about the centre line

$$\begin{aligned} M_f &= \int_0^R \mu p \cdot \frac{2\pi r^2 dr}{\sin \alpha} \\ &= \frac{\mu p \cdot 2\pi}{\sin \alpha} \int_0^R r^2 dr \\ &= \frac{\mu \cdot p \cdot 2\pi \cdot R^3}{\sin \alpha \cdot 3} \\ &= \frac{\mu \cdot W \cdot 2\pi R^3}{\pi \cdot R^2 \cdot 3 \sin \alpha} \\ &= \frac{2\mu \cdot W \cdot R}{3 \sin \alpha} \text{ inch-lb.} \end{aligned} \quad (5)$$

$$\therefore \text{Horse-power absorbed} = \frac{\mu \cdot W \cdot R n}{9900 \sin \alpha} \quad (6)$$

For a truncated cone of maximum and minimum radii R_1 , R_2 respectively, we get

$$\begin{aligned} M_f &= \frac{\mu p \cdot 2\pi}{\sin \alpha} \int_{R_2}^{R_1} r^2 dr \\ &= \frac{\mu \cdot p \cdot 2\pi}{\sin \alpha} \left(\frac{R_1^3 - R_2^3}{3} \right) \\ &= \frac{\mu \cdot W \cdot 2\pi (R_1^3 - R_2^3)}{3\pi (R_1^2 - R_2^2) \sin \alpha} \\ &= \frac{\mu 2W (R_1^2 + R_1 R_2 + R_2^2)}{3 \sin \alpha (R_1 + R_2)} \end{aligned} \quad (7)$$

∴ Horse-power absorbed

$$= \frac{\mu W (R_1^2 + R_1 R_2 + R_2^2) n}{9900 \sin \alpha (R_1 + R_2)} \quad (8)$$

When $\alpha = 90^\circ$ this gives, as we should expect, the same result as for the collar bearing.

Schiele's Pivot.—In the pivots considered up to the present the wear will be greatest at the outside because the velocity is greatest there, thus causing considerable unevenness in wear.

Schiele's pivot is made of such a shape that the wear is theoretically equal if equal pressure be assumed as before.

We will now determine the shape of the curve which will give this result.

Consider an elementary strip (Fig. 80) of the bearing. The load on the strip will be the proportion of W carried by a circular ring of thickness $AC = dr$.

$$Load = \frac{W}{A} \cdot 2\pi r \cdot dr,$$

where A is the horizontal projection of the area over which the weight is carried.

Assuming that the pressure is constant and equal to p we have

$$\therefore p \times AB \cdot 2\pi r \sin \theta = \frac{W}{A} \cdot 2\pi r dr$$

As in the case of the conical pivot we shall have moment of friction on element

$$dM_f = \frac{\mu p \cdot 2\pi \cdot r^2 dr}{\sin \theta}$$

The normal wear ac of the pivot at any strip, in the direction of p , will be proportional to the work done against the friction, if AB is the same for each strip,

because the friction is the force resisting wear; so that work done in wear = friction force \times distance through which it is moved.

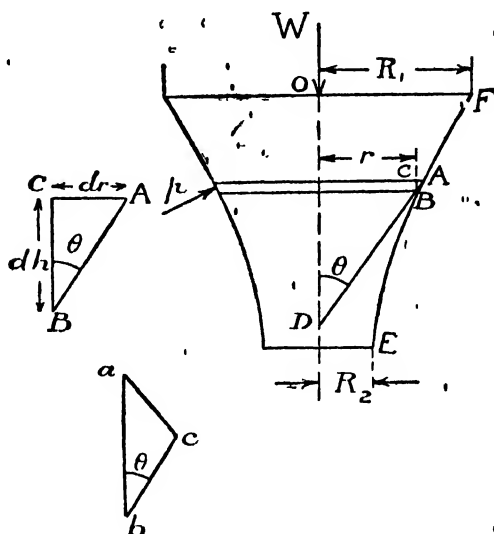


Fig. 80.

Now ab is the vertical wear and if ac is perpendicular to bc we have

$$\sin \theta = \frac{ac}{ab} = \frac{r}{AD}$$

$$\therefore AD = \frac{r \cdot ab}{ac}$$

Now work done against friction per second

$$= \mu p \cdot v = \mu p \cdot 2\pi r n$$

ac is proportional to this, i.e.,

$$ac = k \cdot \mu p \cdot 2\pi r n$$

where k = some constant and since, μ , p , and n are assumed constant we may write

$$\begin{aligned} aq &= K \cdot r \\ \therefore AD &= \frac{r \cdot ab}{Kr} \\ &= \frac{ab}{K} \end{aligned}$$

\therefore if the vertical wear ab of the pivot is constant the length AD is constant.

The curve FAE therefore is such a curve that its tangent AD is of constant length.

This curve is called the tractrix [see p. 221].

§ 58. (E) The Root Mean Square.—It will be readily seen that the mean value of $\sin x$ between the limits $x = 0$ and $x = 2\pi$ is zero, for in the graph the area below the x -axis is equal to the area above it. In many cases a quantity has the same effect whether it is positive or negative, and in such cases one might add on the negative areas as if they were positive and find the mean as before. This would be convenient enough graphically, but would make the integration a little trying. In some cases, however, it is the mean of the square of a quantity that is important, and here no question of negative values arises, since a square is essentially positive.

The heat generated by an electric current is proportional to the square of the current.

Heat generated per second = C^2R ,

where C = current

R = resistance.

If C varies the heat generated in time t is

$$W = \int_0^t R C^2 dt = R \int_0^t C^2 dt.$$

In this case it is the square of the current that we are concerned with, and if we knew *the mean value of C^2* ,

$$\text{Mean } C^2 = K,$$

then the heat generated would be

$$RKt.$$

Suppose \bar{C} is the constant current that generates the same amount of heat in the time t .

Then

$$W = RC^2t = R\int C^2dt,$$

and

$$\bar{C} = \sqrt{\frac{1}{t} \int C^2 dt}$$

which is called the *root mean square current*, is what is important for us to know, especially in the case of *alternating currents*.

For example, suppose an alternating current to be given by

$$C = C_0 \sin \omega t,$$

then the root mean square over one complete period ω .

$$\begin{aligned} \bar{C} &= \sqrt{\left\{ \frac{\omega}{2\pi} C_0^2 \int_0^{2\pi} \sin^2 \omega t dt \right\}} \\ &= \sqrt{\left\{ \frac{\omega}{2\pi} C_0^2 \cdot \frac{1}{\omega} \int_0^{\omega t=2\pi} \sin^2 \omega t d(\omega t) \right\}} \\ &= \frac{C_0}{\sqrt{2\pi}} \sqrt{\left\{ \frac{1}{2} [\omega t - \cos \omega t \sin \omega t]_0^{\omega t=2\pi} \right\}} \\ &\quad \text{(see § 42, 5° (2).)} \\ &= C_0 \sqrt{\frac{1}{2} \frac{(2\pi)}{2\pi}} \\ &= \frac{1}{\sqrt{2}} C_0. \end{aligned}$$

GRAPHICAL METHOD.—Graphically the root mean square could be determined by constructing the graph of C^2 from a knowledge of C and then proceeding as before, but there is a better method.

Instead of determining the position of a point P (Fig. 81) by means of its rectangular co-ordinate (x, y)

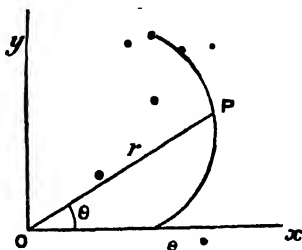


FIG. 81.

we may use the *polar* co-ordinates

$$r = OP, \theta = \angle xOP.$$

A relation between r and θ will determine a curve as before, for example the curve (a part of it)

$$r = e^{\theta} \quad (A)$$

is shown in Fig. 81. To trace this curve we measure along the radius OP corresponding to each angle θ , a length given by the equation (A).

THE "AREA OF THE CURVE," i.e. of the sector OAP (Fig. 82) is now considered as the sum of the triangles $OAP_1, OP_1P_2, OP_2P_3, \dots$ approximately.

Now $OP = r$, $OQ = r + \delta r$, $\angle POQ = \delta\theta$; so the area of the triangle

§ 59. (C) Problems on the Flow of Water.—
A number of problems upon the flow of water introduce the operation of integration.

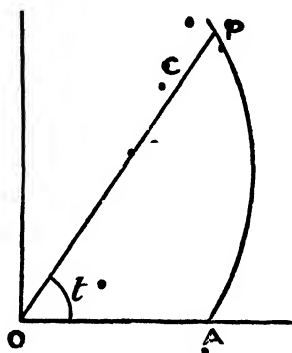


FIG. 82a.

FLOW OVER A RECTANGULAR NOTCH OR WEIR.—
Consider an elementary strip of water at depth h (Fig. 83) below the surface. This will possess a

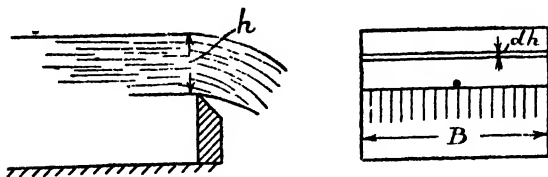


FIG. 83.

velocity due to falling a height h ,

$$\text{i.e. } v = \sqrt{2gh}.$$

\therefore flow of water through element per second = C_d
 \times area of strip \times velocity, where C_d is a constant
called the coefficient of discharge

\therefore elementary flow

$$\begin{aligned}
 &= dQ = C_d \times \text{area of strip} \times \text{velocity} \\
 &= C_d \cdot B dh \cdot \sqrt{2gh} \\
 &= \sqrt{2g} \cdot C_d \cdot B h^{\frac{3}{2}} dh \\
 \therefore \text{total flow } Q &= \int_0^H \sqrt{2g} \cdot C_d B h^{\frac{3}{2}} dh \\
 &= \sqrt{2g} \cdot C_d B \int_0^H h^{\frac{3}{2}} dh \\
 &= \frac{2}{3} \sqrt{2g} C_d B \left[h^{\frac{3}{2}} \right]_0^H \\
 &= \frac{2}{3} C_d \cdot B H \sqrt{2gH}.
 \end{aligned}$$

TRIANGULAR NOTCH OR WEIR.—In this case (Fig. 84) we have as before

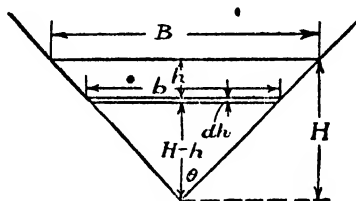


FIG. 84.

$$dQ = \sqrt{2g} \cdot C_d \cdot b h^{\frac{3}{2}} dh.$$

Now $b = 2(H - h) \tan \theta$

$$\begin{aligned}
 \therefore dQ &= \sqrt{2g} \cdot C_d \cdot 2 \tan \theta (H - h) h^{\frac{3}{2}} dh \\
 \therefore Q &= \sqrt{2g} \cdot C_d \cdot 2 \tan \theta \int_0^H (H h^{\frac{3}{2}} - h^{\frac{5}{2}}) dh \\
 &= \sqrt{2g} C_d \cdot 2 \tan \theta \cdot \left[\frac{2}{3} H h^{\frac{3}{2}} - \frac{2}{5} h^{\frac{5}{2}} \right]_0^H \\
 &= \sqrt{2g} \cdot C_d \cdot 2 \tan \theta \cdot \frac{4}{15} H^{\frac{5}{2}} \\
 &= \frac{8}{15} C_d \tan \theta H^{\frac{5}{2}} \sqrt{2gH} \\
 \text{or } &= \frac{4}{15} C_d B H \sqrt{2gH} \quad (2)
 \end{aligned}$$

TIME TO EMPTY A RECTANGULAR TANK WITH A

SMALL ORIFICE.—Let the area of the orifice be a (Fig. 85) and that of the tank [in plan] be A .

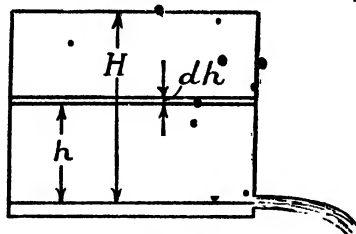


FIG. 85.

Let the head of water at any instant be h and let its surface descend through a distance dh in an element of time dt . Then velocity of flow through orifice.

$$= v = \sqrt{2gh}$$

\therefore flow through orifice in time dt

$$= C_d \times \text{area of orifice} \times \text{velocity} \times dt$$

$$= C_d \cdot a \sqrt{2gh} \cdot dt$$

but the flow through the orifice must be equal to the diminution of volume in the tank, i.e. $A dh$

$$\therefore A dh = C_d a \sqrt{2gh} dt$$

$$\therefore dt = \frac{A dh}{\sqrt{2g} \cdot C_d \cdot a \cdot h^{\frac{1}{2}}}$$

\therefore time required to empty tank $= t$

$$= \int_H^0 \frac{A}{\sqrt{2g} \cdot C_d \cdot a \cdot h^{\frac{1}{2}}} dh$$

$$= \frac{A}{\sqrt{2g} \cdot C_d \cdot a} \int_H^0 h^{-\frac{1}{2}} dh$$

$$= \frac{A}{\sqrt{2g} \cdot C_d \cdot a} \left[2h^{-\frac{1}{2}+1} \right]_H^0$$

$$= \frac{2A}{\sqrt{2g} \cdot C_d \cdot a} \sqrt{H} = \frac{A}{a C_d} \sqrt{\frac{2H}{g}} \quad (3)$$

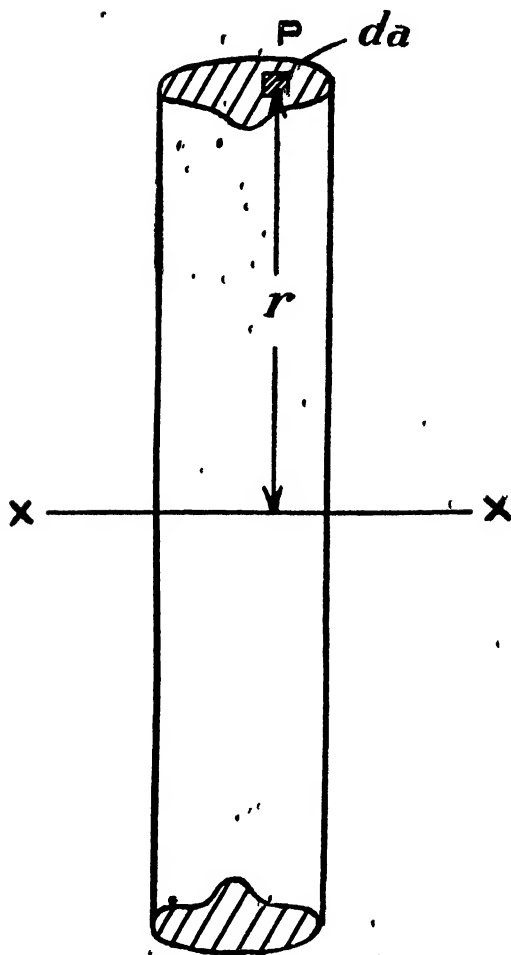


FIG. 86.

If the tank is of a different shape the method will be the same but the area A will not be a constant, and so the integration will be rather more complicated.

§ 60. (M) **Kinetic Energy Stored in a Flywheel.**—Consider an element da of area situated at a point P (Fig. 86) in the cross section of a flywheel rotating at n revolutions per second.

Then if ρ is the density of the material [weight of unit volume] the mass of the ring of which da is the section will be equal to

$$dM = \frac{\text{weight of ring}}{g} = \frac{\rho \cdot 2\pi r da}{g}$$

and it has a velocity $2\pi rn$

\therefore kinetic energy of element $= dE = \frac{1}{2} \text{ mass} \times \text{velocity}^2$

$$\begin{aligned} &= \frac{1}{2} dM \cdot (2\pi rn)^2 \\ &= 2\pi^2 n^2 r^2 dM \end{aligned}$$

\therefore total K. E. of flywheel $= E = 2\pi^2 n^2 \int r^2 dM$
but $\int r^2 dM$ is what we have called the *polar moment of inertia* of the flywheel $= J$

$$\therefore \text{K. E.} = 2\pi^2 n^2 J \quad (1)$$

It is common to express this in terms of ω , the angular velocity which is equal to $2\pi n$

$$\text{then K. E.} = \frac{1}{2} J \omega^2 \quad (2)$$

If k is the polar radius of gyration of the flywheel,

$$J = Mk^2 = \frac{Wk^2}{g}$$

$$\therefore \text{then K. E.} = \frac{1}{2} M \omega^2 k^2 = \frac{\frac{1}{2} W \omega^2 k^2}{g} \quad (3)$$

§ 61. (C) **The Deflections of Beams.**—It can be shown that in a beam the relation holds that the

slope θ of the beam at any point is given when such slope is small by the relation

$$\tan \theta = \int \frac{M dx}{EI} \quad (1)$$

where M = the bending moment at any point .

E = Young's Modulus

I = M. I. of cross section about neutral axis.

and the deflection y of the beam is given by

$$y = \int \tan \theta dx \quad (2)$$

To get the deflection y therefore from the bending moment we have to integrate twice; this is written

$$y = \iint \frac{M dx^2}{EI} \quad (3)$$

As a simple example of this take the case of a beam of span l simply supported at the ends and carrying a uniformly distributed load of p units per unit length (Fig. 87).

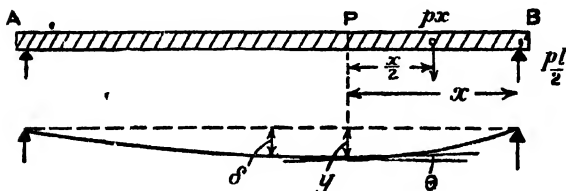


FIG. 87.

Consider any point P along the beam.

The forces acting to the right are an upward reaction equal to $\frac{pl}{2}$ at the end B and a downward force px acting at the centre of the length PB .
 \therefore moment of forces to the right of B

$$\begin{aligned} \therefore \text{bending moment at P} = M &= \frac{pl \cdot x}{2} - \frac{px \cdot x}{2} \\ &= \frac{p}{2} (lx - x^2) \end{aligned} \quad (4)$$

$$\therefore \text{from (1) } \tan \theta = \int_0^x \frac{p}{2} \frac{(lx - x^2)}{EI} dx$$

\therefore if E and I are constant

$$EI \tan \theta = \frac{p}{2} \left(\frac{lx^2}{2} - \frac{x^3}{3} \right) + C_1 \quad (5)$$

Now from symmetry the slope of the beam is zero at the centre where $x = \frac{l}{2}$

$$\therefore 0 = \frac{p}{2} \left(\frac{l \cdot l^2}{8} - \frac{l^3}{24} \right) + C_1$$

$$\text{i.e. } C_1 = -\frac{pl^3}{16} \left(1 - \frac{1}{3} \right) = -\frac{pl^3}{24}$$

$$\therefore EI \tan \theta = \frac{plx^2}{4} - \frac{px^3}{6} - \frac{pl^3}{24} \quad (6)$$

\therefore from equation (2)

$$\begin{aligned} EI \cdot y &= \int EI \tan \theta \cdot dx \\ &= p \int \left(\frac{lx^2}{4} - \frac{x^3}{6} - \frac{l^3}{24} \right) dx \\ &= p \left(\frac{lx^3}{12} - \frac{x^4}{24} - \frac{l^3 x}{24} \right) + C_2 \end{aligned} \quad (7)$$

Now $y = 0$ at the end where $x = 0 \therefore C_2 = 0$

$$\therefore EI \cdot y = \frac{px}{12} \left(lx^2 - \frac{x^3}{2} - \frac{l^3}{2} \right) \quad (8)$$

The maximum deflection δ occurs at the centre where $x = \frac{l}{2}$

$$\begin{aligned}
 \therefore EI \cdot \delta &= \frac{pl}{24} \left(\frac{l^3}{4} - \frac{l^3}{16} - \frac{l^3}{2} \right) \\
 &= \frac{pl^4}{2 \times 24} \left(\frac{1}{2} - \frac{1}{8} - 1 \right) \\
 &= \frac{pl^4}{24} \left(-\frac{5}{8} \right) \\
 &= -\frac{5pl^4}{384} \\
 \therefore \delta &= -\frac{5pl^4}{384 EI} \quad (9)
 \end{aligned}$$

The minus sign indicates that the deflection is downwards.

The deflections of beams loaded and supported in various other manners are obtained in a similar way and will be found in the various text-books dealing with the subject.

EXERCISES 7.

1. Show that the centroid of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is at distances $\frac{4a}{3\pi}$ and $\frac{4b}{3\pi}$ from the axes.

2. A triangle has its vertex at the surface of a liquid and has its base horizontal. If h is the height of the triangle, find the depth of its centre of pressure.

3. The pressure of one pound of saturated steam at 347°F. is 130 lb. per sq. in., and the volume is 3.44 cub. ft. Find the work done in an adiabatic expansion which doubles the volume, $\gamma = 1.135$. (M.)

4. Find the work done in suddenly compressing one pound of dry air originally at 32°F. to $\frac{1}{4}$ th of its original volume. Use the equations $PV = 53.18 \text{ T}$ (abs) and $PV^{1.404} = \text{constant}$.

P = pressure in lbs. per sq. ft.

V = Volume in cu. ft.

Thus get the temperature after expansion and apply the formula: work done $= \frac{53.18 (T_1 - T_2)}{404}$. (M.)

5. A certain volume of air is drawn into a cylinder at a pressure 120 lb. per sq. in., and is expanded isothermally until the volume is 6 times as great. Find the mean pressure during inlet and expansion. (M.)

6. Find the mean pressure under the conditions of the previous question if the expansion is adiabatic taking $\gamma = 1.408$. (M.)

7. Find the area included between the axis of x and the curve $9y = x^2(x + 3)$, x and y being in inches.

8. Find the co-ordinates of the centroid of the area included between the curve $y = \frac{a^2}{a^2 + x^2}$ and the axis of x .

9. Find the centroid of the area included between one semi-undulation of the curve $y = 6 \sin \frac{x}{2}$ and the axis of x .

10. Find the centroid of the circular spandril formed by the quadrant of a circle of radius r and the tangents at its extremities.

11. A trapezium of depth h has one side a of its parallel sides a, b , in the surface of a liquid. Prove that the depth of the centre of pressure is

$$\frac{a + 3b}{2(a + b)} \cdot h.$$

12. Find the radius of gyration of a uniform circular disk of radius r about an axis normal to its plane passing through a point on its circumference. (C.)

13. A bending moment diagram of a beam of span l is made up of a triangle of height $\frac{pl^2}{16}$ with apex at

the centre and a parabola, extending from the right-hand end to the centre, of height $\frac{p^2}{32}$. Find the position of the centroid of the diagram. (C.)

14. The moment of inertia of a channel section about its base is 15.639 inch units, and the centroid is 1.185 inches from the base. If the area of the section is 5.219 sq. in., find the moment of inertia and radius of gyration about a line through the centroid parallel to the base. (C.)

15. If the root mean square value of an alternating electric current is ten amperes, what is the maximum value? (E.)

16. What will be the *average* value of the current in one semi-alternation in the previous exercise? (E.)

17. Find by means of the theorem of Pappus the distance from the diameter of the centroid of (a) the surface of a hemisphere, (b) a solid hemisphere.

18. A rectangular beam 12 ft. long, 1 ft. deep and 5 in. wide rests on supports at its ends. What uniformly distributed load will cause a deflection equal to $\frac{1}{1000}$ of its span. $E = 1.8 \times 10^6$ lb. per sq. in. (C.)

19. A cast-iron pipe of 18 in. internal diameter and 1 in. thick, rests on supports 40 feet apart. Find the maximum bending stress at the centre when the pipe is full of water. Take weight of cast iron = 450 lb. per cub. ft. and of water 62.3 lb. per cub. ft. (C.)

20. The resilience or work done in bending of a beam is given by $\int_0^L \frac{M^2 dx}{2EI}$ where E and I are constant. If $M = \frac{w}{2}(Lx - x^2)$ as is the case for uniformly distributed loading, find the resilience. (C.)

21. The entropy ϕ of a substance is defined by $\phi = \int \frac{dQ}{T}$ where Q is the heat absorbed and T is the

absolute temperature. If the specific heat of water is taken as 1, the entropy above freezing point is $\int_{T_1}^{T_2} \frac{dT}{T}$; find the entropy of one pound of water at 300°F. ($761^\circ \text{F. absolute}$). (M)

22. If the specific heat of superheated steam is .48, find the entropy of a pound of steam at 300°F superheated to 390°F . The latent heat L at this temperature (300°F.) may be taken as 903 so that the formula for the entropy becomes

$$\phi = \int_{T_1}^{T_2} \frac{dT}{T} + \frac{L}{T_1} + .48 \int_{T_1}^{T_2} \frac{dT}{T}$$

It is $\frac{L}{T}$ during the evaporation period, because T is constant and $\int dQ = L$. (M)

23. A vertical shaft 4 inches in diameter turns on a flat pivot. The weight of the shaft with its various fittings is 2500 lb. If $\mu = .08$ and the speed is 140 revolutions per min., find the horse-power used in friction by the pivot. (E) (M)

24. In the above question find the horse-power lost if the pivot were a cone frustum, the cone angle being 60° and the least diameter $1\frac{1}{2}$ inches (E, M)

25. Will the work absorbed in friction in a Schiele pivot be less than or more than that of a truncated cone of the same outer radius, the half angle of cone being 45° and the angle of the tangent to tractrix at the longer radius being 45° ? The radius of the small end of the conical pivot may be taken as one half the large radius. (E, M.)

26. Two tanks whose surface areas A are equal to each other have a difference in water level of H and are connected by a small orifice O , below the level of the water in either and of area a . If C_d be the coefficient of discharge, find the time from the opening of the orifice until the level in the two tanks is equal (note that the change in head is twice eitherise or fall). (C.)

27. A large rectangular orifice of breadth b and

depth h is placed in the side of a tank. If H is the head of water above the top of the orifice show that the flow is $Q = \frac{2}{3}bC_d \sqrt{2g} (H \sqrt{1-h} - H^{\frac{3}{2}})$.

28. Find the time to empty a rectangular chamber 120 ft. square containing 15 ft. depth of water, which is allowed to flow out through a rectangular orifice 2 ft. by 1 ft., the top of which is level with the floor of the chamber, take $C_d = .62$. (C.)

29. A tank 10 ft. square and 10 ft. deep has a circular orifice 4 inches in diameter in the bottom. If the tank is filled and the water turned off, how long will it take to empty taking $C_d = .62$? (C.)

30. A hemispherical cistern is 20 ft. in diameter, and it is full of water. How many minutes will it take to lower the depth of the water 5 ft., if the water escapes through a 3 inch diameter sharp-edged hole in the bottom of the cistern. Take the coefficient of discharge as .60. (C.)

31. Calculate the discharge through a 90° angular notch, the height of the still water surface above the bottom being 15.5 inches; take .60 as the coefficient. (C.)

32. A tube of length $2r$, closed at both ends and full of liquid, revolves in a horizontal plane about an axis, at right angles to and bisecting its own axis. Prove that the intensity of pressure on its end will be $\frac{w\omega^2 r^2}{2g}$ where w is the weight per unit volume of the liquid and ω is the angular velocity. (C.)

33. A cantilever is fixed horizontally at one end and is free at the other end, the length being l . Assuming that the deflections y are given by $\frac{d^2y}{dx^2} = \frac{M}{EI}$ and M at a distance x from the free end is Wx , find the deflection at the free end. (C.)

34. In arch problems the quantity $\int y^2 dx$ is used in determining the thrusts. For a parabolic arch $y = \frac{4x}{l^2} (lx - x^2)$; find $\int_0^l y^2 dx$ for such an arch. (C.)

35. The following integration also occurs in problems on parabolic arches $\int_{-L}^L (L^2x - 2Lx^2 + x^3) dx$.

Evaluate it. (C.)

ANSWERS TO EXERCISES.

EXERCISES 7.

2. $\frac{3h}{4}$ 3. 42200 ft.-lb. 4. 7980 ft.-lb.
5. 55.8 lb. per sq. in. 6. 45.4 lb. per sq. in.
7. .75 sq. in. 8. $0\frac{a}{4}$ 9. $\frac{3\pi}{4}$ 10. .223 r. 12. $\frac{\sqrt{6}r}{2}$
13. $\frac{7l}{16}$ from the right-hand end. 14. $I = 6.316$
inch units, $k = 1.01$ in. 15. 14.14 amperes.
16. 9.01 amperes. 17. (a) Surface = $4\pi r^2 = \text{arc} \times \text{path}$
of centroid = $\pi r \times 2\pi r \therefore = \frac{2r}{\pi}$, (b) $\frac{4r}{3\pi}$.
18. 4800 lb. 19. 2640 lb. per sq. in. 20. $\frac{wL^5}{240EI}$
21. $\log_e \frac{761}{493} = .432$. 22. $\log_e \frac{761}{493} + \frac{903}{761}$
 $+ .48 \log_e \frac{851}{761} = 1.67$. 23. .59. 24. 1.32.
25. Schiele = $\mu WR_1 \sqrt{2}$ and conical $\frac{\mu WR_1 7 \sqrt{2}}{9}$ which
is less. 26. $t = \frac{\Lambda \sqrt{II}}{Cda \sqrt{2g}}$ 28. 3 hrs. 7 min.
nearly. 29. 25 min. nearly. 30. $\frac{37}{Wl^3}$ min. 19 sec.
31. 4.87 cub. ft. per second. 33. $\frac{8r^2L}{EI^3}$ 34. 15
35. $\frac{L^4}{12} (1 - a) (1 + a - 5a^2 + 3a^3)$.

CHAPTER VIII.

PARTIAL DIFFERENTIATION.

§ 62. **Several Independent Variables.**—A physical quantity may depend upon several other quantities which are independent of each other, instead of depending upon one only. For instance, the volume v of a given mass of a perfect gas is connected with the pressure p and the (absolute) temperature θ in the following way :—

$$pv = R\theta, \quad (1)$$

where R is a constant. So

$$v = R \frac{\theta}{p} \quad (2)$$

The temperature and pressure may here be adjusted independently of each other. We might vary the temperature θ alone, keeping the pressure constant; then the rate of change of volume with temperature would be got by differentiating equation (2),

$$\frac{\partial v}{\partial \theta} = \frac{R}{p}.$$

Or again we might keep θ constant and vary the pressure; the rate of change of the volume with the pressure would be

$$\frac{\partial v}{\partial p} = -\frac{R\theta}{p^2}.$$

When a quantity z depends upon two quantities x and y , then the rate of change of z with respect to x when y is supposed to remain constant is called a partial differential coefficient. It is written $\frac{\partial z}{\partial x}$ (pronounced *day z over day x*).

To take another example, suppose

$$z = 2x^2 + 3xy + y^2 + 2x + 1$$

then

$$\frac{\partial z}{\partial x} = 4x + 3y + 2$$

$$\frac{\partial z}{\partial y} = 3x + 2y.$$

Or again let

$$z = \tan^{-1} \frac{y}{x}$$

then

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \times -\frac{y}{x^2} = -\frac{y}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2 + y^2}.$$

HIGHER DIFFERENTIAL COEFFICIENTS.—We can as before find the slope of the slope with respect to x , i.e. for the first of the above examples

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} (4x + 3y + 2) = 4 \text{ (a constant).}$$

And similarly

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} (3x + 2y) = 2.$$

But we have in this case another second order slope, viz. the slope with regard to y of the slope with regard to x ,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (4x + 3y + 2) = 3.$$

And also

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} (3x + 2y) = 3.$$

It will be noted that in this case

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

This is always the case for ordinary laws. For our second example

$$z = \tan^{-1} \frac{y}{x}$$

we have

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{-y}{x^2 + y^2} \right) = -y \frac{\partial}{\partial x} (x^2 + y^2)^{-1} \\ &\quad \times \frac{\partial}{\partial x} (x^2 + y^2) \end{aligned}$$

$$= -y \times (-1) (x^2 + y^2)^{-2} \times 2x = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = \frac{(x^2 + y^2) \frac{\partial x}{\partial x} - x \frac{\partial}{\partial x} (x^2 + y^2)}{(x^2 + y^2)^2} \\ &= \frac{(x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \end{aligned}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{-y}{x^2 + y^2} \right) = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

§ 63. Differentials and Small Variations.—

SMALL VARIATIONS.—To return to one of our earliest examples, suppose x represents the distance covered by a motor-car at a given instant and t represents the time taken in covering this distance. The speed

is $\frac{dx}{dt}$: so the distance covered in a small interval of time following δt will be

$$\frac{dx}{dt} \times \delta t$$

approximately; this is not exact because $\frac{dx}{dt}$ is the speed at the beginning of the interval δt , and the speed is varying, but if δt is very small the variation in the speed will likewise be small, and hence the error will be negligible.

To be more precise, let us suppose that the car is moving with a speed of 40' per second and has a uniform acceleration of 1' per second per second; let the interval of time $\delta t = \frac{1}{10}$ second. Then the quantity which approximates to the distance covered

$$\frac{dx}{dt} \delta t = 40' \times \frac{1}{10} = 4',$$

a "small quantity". But the average speed during the interval is $40\frac{1}{20}'$ per second, and the exact distance covered is

$$40\frac{1}{20} \times \frac{1}{10} = 4\frac{1}{200}'.$$

The error made is therefore $\frac{1}{200}'$, i.e. only $\frac{1}{200}$ of 4', and this we call a "small quantity of second order". The proportional error is still less if we take a shorter interval, say $\frac{1}{100}$ second. In this case

$$\frac{dx}{dt} \delta t = 40' \times \frac{1}{100} = 0\cdot4'.$$

The average speed is $40\frac{1}{100}$, and the exact distance covered is $(0\cdot4' + \frac{1}{20000})$. The error $\frac{1}{20000}'$ is now only $\frac{1}{20000}$ of 0\cdot4'.

Thus in general the variation of any quantity y in terms of x may be taken as

$$\delta y = \frac{dy}{dx} \delta x$$

when δx is small, and the smaller δx is the closer is the approximation.

These small variations

$$\delta x = \text{variation in } x$$

$$\frac{dy}{dx} \delta x = \text{variation in } y \text{ corresponding,}$$

are generally called *differentials*.

Thus if

$$y = x^3$$

the differential of y is

$$\delta y = 3x^2 \delta x.$$

We obtain small variations similarly when the dependent quantity varies with several others. The relation between pressure, volume and temperature for a given mass of perfect gas is

$$v = R \frac{\theta}{p} \text{ (see § 62).}$$

A variation in v when θ varies, p being constant, is

$$\delta_p v = \frac{R}{p} \delta \theta;$$

and a variation in v when p varies, θ being constant is

$$\delta_\theta v = - \frac{R\theta}{p^2} \delta p.$$

TOTAL VARIATION.—What will be the variation in v in the last example when both θ and p vary? The answer is that instead of supposing θ and p to vary simultaneously, we can suppose that θ varies first to its new value $\theta + \delta\theta$, p being constant, and that p then varies to its new value $p + \delta p$, θ remaining constant at $\theta + \delta\theta$. Then the sum of these two

variations' is the total variation in v . The first variation in v is

$$\delta_p v = \frac{R}{p} \delta \theta.$$

The second variation in v , i.e. with the temperature at $\theta + \delta \theta$, will be very nearly what it would be at θ (because $\delta \theta$ is small). Thus

$$\delta_{\theta} v = - \frac{R(\theta + \delta \theta)}{p^2} \delta p = - \frac{R\theta}{p^2} \delta p.$$

Finally the total variation is

$$\begin{aligned} \delta v &= \delta_p v + \delta_{\theta} v \\ &= \frac{\partial v}{\partial \theta} \delta \theta + \frac{\partial v}{\partial p} \delta p \\ &= \frac{R}{p} \delta \theta - \frac{R\theta}{p^2} \delta p, \end{aligned}$$

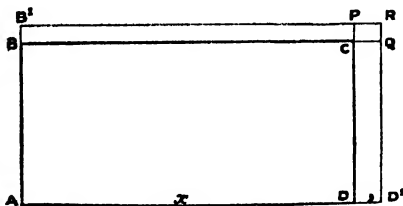


FIG. 88.

very nearly, neglecting small quantities of second order.

In general if z depends upon x and y ,

$$\delta z = \frac{\partial z}{\partial x} \delta x + \frac{\partial z}{\partial y} \delta y.$$

Let us take a simple case to see more clearly the kind of error we are neglecting. The area of a rectangle whose sides are x and y is

$$z = xy$$

Then according to our work,

$$\delta z = \frac{\partial z}{\partial x} \delta x + \frac{\partial z}{\partial y} \delta y = y \delta x + x \delta y.$$

Now δz is really the increase in the area of the rectangle when the side AD increases by $DD' = \delta x$, and the side AB increases by $BB' = \delta y$. The quantity $y \delta x$ is the area $DD'QC$, while $x \delta y$ is $BCPB'$. So the quantity neglected is here the area $CQRP$, and this is small compared with the total increase in

EXERCISES 8.

1. If $u = x^n y^m$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.
2. If $z = (x + y)^2$, show that $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$.
3. If $r = \sqrt{x^2 + y^2}$, find $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$.
4. If $ax^2 + 2bxy + by^2 + 2gx + 2fy + C = 0$ is the general equation to the series of curves known as "conics" find the slope of the tangent (i.e. $\frac{dy}{dx}$).
5. Calculate the difference for one minute in a table of \log_{10} sines in the neighbourhood of 45° . Proceed as follows: $y = \log_{10} \sin x$, $\delta y = \delta x \frac{d \log_{10} \sin x}{dx}$

$$= \frac{\pi}{60 \times 180} \frac{d \log_{10} \sin x}{dx} = \frac{\pi}{2.3 \times 60 \times 180} \frac{d \log_e \sin x}{dx}.$$
6. The height of a tower is calculated by measuring a length of 100 ft. from the base and measuring the angle of elevation which comes to 30° . If the error in the angle be one minute, find the error in the computed height.
7. In a tangent galvanometer the current is proportional to the tangent of the inclination of the

needle. Find the portion of the scale where the proportional error in the current due to a given error of reading is least.

8. The height of a cliff is calculated by finding that a stone takes 3 seconds to reach the bottom. If the error in timing is $\frac{1}{10}$ second, what will be the error in the calculated height?

9. If w is the weight of a body in air and w' is its weight immersed in water, the specific gravity of the body is given by $\frac{w}{w - w'}$. If the weight in water is correct and that in air has a small error e , what will be the resulting error in the calculated specific gravity?

10. Take the same problem as before but take each measurement as liable to errors e_u and e_w respectively. If $w = 120.5$ gm. and $w' = 110.9$ gm., $e_u = 3$ mg. and $e_w = 4$ mg. find the total possible percentage error in the calculation.

ANSWERS TO EXERCISES.

EXERCISES 8.

1. $ny x^{n-1}, mx^m y^{n-1}$. 3. $\frac{x}{r}, \frac{y}{r}$. 4. $-\frac{ax + by + g}{hx + by + f}$
5. .000126. , 6. .47 in. 7. 45. 8. about 10 ft.
9. $\frac{ew'}{(w - w')^2}$. 10. .07 per cent.

CHAPTER IX.

DIFFERENTIAL EQUATIONS.

§ 64. **Some Examples.**—A differential equation is a relation between a variable quantity and one or more of its differential coefficients.

1°. The current C in a circuit whose resistance is R and whose inductance is L satisfies the relation

$$RC + L \frac{dC}{dt} = V$$

where V is the potential difference at the two extremities.

This is a differential equation of *first order*, since the first differential coefficient $\frac{dC}{dt}$ only occurs in it.

The independent quantity is the time t . The potential V we may suppose known and it then has an expression in terms of t . The coefficients R and L are simply constants.

2°. The acceleration of a body falling freely is given by

$$\frac{d^2x}{dt^2} = g = 32 \text{ in ft.-sec. units}$$

where x is the distance through which the body has fallen in time t . This is a differential equation of

second order. It will be noted that neither x nor $\frac{dx}{dt}$ occur in this equation.

3°. Suppose a point moves in a straight line so that it always has an acceleration towards a point in this line proportional to its distance from the point. Then its acceleration

$$\frac{d^2x}{dt^2} = -kx$$

where k is a constant.

4°. To take a more general case, suppose the particle is urged towards the point with another given periodic force $= mb \sin pt$ (period $= 2\pi/p$, mass $= m$), and that there is also a resistance proportional to the velocity (to $\frac{dx}{dt}$). Then the mass \times acceleration = acting force;

$$m \frac{d^2x}{dt^2} = -mkx + mb \sin pt - m\mu \frac{dx}{dt}$$

or

$$\frac{d^2x}{dt^2} + \mu \frac{dx}{dt} + kx = b \sin pt.$$

If the right-hand side is absent, this is the relation satisfied by a *damped harmonic vibration*. With the right-hand side it is a *damped forced harmonic vibration*.

To solve a differential equation means to find an expression for the dependent quantity which satisfies it. We do not propose to give a systematic discussion of methods, but we will give the solutions for one or two important cases.

In the same way as in the case of integration, there is no direct method of solving a differential

equation, we can only solve it by reducing it to some standard form of which the solution can be recognized.

§ 65. **Solutions.**—The solution of a differential equation involves an integration. Take the equation

$$\frac{dx}{dt} = k t^2.$$

This expresses that the velocity of a particle is proportional to the square of the time, and we want to find an expression for the displacement.

Evidently

$$x = k \int t^2 dt = \frac{1}{3} k t^3 + C.$$

This is the law for some value of C .

To determine this value we have an *initial condition*: for example, initially $x = 1$. Putting $t = 0$, $x = 1$ we get

$$1 = 0 + C$$

which determines C :

$$C = 1.$$

Hence

$$x = \frac{1}{3} k t^3 + 1.$$

1° (E) Take next a simple case of 1° (§ 64), namely the case where $V = 0$; this means that the E.M.F. is cut out of a circuit which is then left to itself.

$$RC + L \frac{dC}{dt} = 0$$

or

$$\frac{dC}{dt} = -\frac{R}{L} C.$$

We recognize at once that the law is exponential,¹ and as a matter of fact it is

$$C = C_0 e^{-\frac{R}{L} t}$$

as can be readily verified by substituting, for

¹ See p. 63.

$$\begin{aligned}\frac{dC}{dt} &= C_0 \times \left(-\frac{R}{L}\right) \times e^{-\frac{R}{L}t} \\ &= -\frac{R}{L} \cdot C_0 e^{-\frac{R}{L}t} = -\frac{R}{L} C_0,\end{aligned}$$

Here C_0 is the value of C when $t = 0$.

When V is a constant, the solution of the equation quoted above is

$$C = K e^{-\frac{R}{L}t} + \frac{V}{R}$$

where K is some constant. [Verify this by substitution.] For example if an E.M.F. equal to V is suddenly applied to a circuit in which there is initially no current, then $C = 0$ when $t = 0$; hence

$$0 = K \times 1 + \frac{V}{R} \text{ or } K = -\frac{V}{R}$$

and the formula for C is

$$C = \frac{V}{R} \left(1 - e^{-\frac{R}{L}t}\right).$$

Note that the second term $e^{-\frac{R}{L}t}$ soon becomes very small, so that C quickly approaches its limiting value,

viz. $\frac{V}{R}$.

2°. We take next the equation

$$\frac{d^2x}{dt^2} = g.$$

At once, on integration,

$$\frac{dx}{dt} = gt + A,$$

where A is a constant. Suppose initially the velocity

$\frac{dx}{dt}$ (downwards) = u ; then

$$u = 0 + A \text{ or } A = u$$

and

$$\frac{dx}{dt} = gt + u,$$

Integrate again

$$x = g \cdot \frac{1}{2} t^2 + ut + B,$$

where B is a constant. Suppose that we measure distances from the starting-point of the body; then

$$x = 0 \text{ when } t = 0,$$

so that

$$0 = 0 + 0 + B, \quad B = 0,$$

and we get finally the familiar formula for the displacement of a body moving with a constant acceleration g ,

$$x = ut + \frac{1}{2}gt^2.$$

SIMPLE HARMONIC MOTION.—3'. (M) We take next

$$\frac{d^2x}{dt^2} = -m^2x,$$

where we have written m^2 for k .

Note that

$$\frac{d}{dt} \sin t = \cos t$$

$$\text{and } \frac{d^2}{dt^2} \sin t = -\sin t.$$

So that the second differential coefficient of $\sin t$ is $\sin t$ reversed in sign. This also applies to $\cos t$. This gives us a clue to a solution;

$$x = \sin mt$$

$$\text{or } x = \cos mt$$

would both do. If we combine them putting in constants as coefficients, we get

$$x = A \sin mt + B \cos mt \quad (1)$$

and by substitution we can easily verify that this satisfies the equation.

Let us take a specific case.

A point moves with an acceleration μx towards a

point 0, starting at rest from an initial distance d , where x is the distance at any time t . Then

$$\text{when } t = 0, x = d \text{ and } \frac{dx}{dt} = 0$$

(i.e. velocity is zero). Substitute in (1).

$$d = A \sin m \cdot 0 + B \cos m \cdot 0$$

$$\text{or } d = 0 + B \cdot 1$$

$$\text{i.e. } B = d.$$

Now differentiate (1)

$$\frac{dx}{dt} = Am \cos mt - Bm \sin mt.$$

Substituting the initial values

$$0 = Am \cdot 1 - Bm \cdot 0$$

$$\text{whence } Am = 0 \text{ or } A = 0$$

and finally

$$x = d \cos mt$$

$$\frac{dx}{dt} = -dm \sin mt.$$

This is a case of simple harmonic motion (§§ 6, 16).

In this equation $x = d$ when $mt = 0$ and again equals d when $mt = 2\pi$

$$\therefore t = \frac{2\pi}{m}$$

is the time period between two successive identical positions or in the case of oscillation is the time of a double swing.

The most convenient form of this result is

$$t = 2\pi \sqrt{\frac{\text{Mass}}{\text{Force at unit distance}}}$$

and will be understood from the following:—

$$\frac{d^2x}{dt^2} = \text{acceleration } (a)$$

$$\therefore \text{when } x = 1, a = -m^2$$

(we take $x = -1$, not $x = +1$, so that the acceleration towards the origin may be positive)

$$\therefore m = \sqrt{a}$$

$$\text{also } \frac{1}{a} = \frac{\text{mass}}{\text{force}}$$

$$\therefore \frac{1}{m} = \sqrt{\frac{\text{Mass}}{\text{Force to cause unit displacement}}}$$

EULER'S[†] FORMULA FOR LONG STRUTS AND COLUMNS.—(4). (C) Although the strength of columns has no apparent connexion with simple harmonic motion, it is as a matter of fact derived from the same form of differential equation.

Referring to Fig. 89, suppose that a strut is so long that the direct stress when deflected is negligible compared with the stress due to bending.

Now suppose that for some reason or other the column becomes slightly deflected (the deflection is shown exaggerated in the figure); then the bending moment at any point Q will be equal to

$$M_Q = P \cdot y \quad \dots \dots \dots (1)$$

For a beam with a small deflection we may write

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = - \frac{Py}{EI}$$

$$\text{i.e. } \frac{d^2y}{dx^2} = - \frac{P \cdot y}{EI} \quad \dots \dots \dots (2)$$

Putting $\frac{P}{EI} = m^2$, this is the same general equation as that we have just considered and the general solution is

$$y = A \sin mx + B \cos mx \quad \dots \dots \dots (3)$$

* The sign is minus because a counter clockwise moment across the section, i.e. a positive moment, gives a negative value of $\frac{d^2y}{dx^2}$, i.e. causes the slope to decrease.

Now we see from the figure that y has the same value for equal positive and negative values of x , and

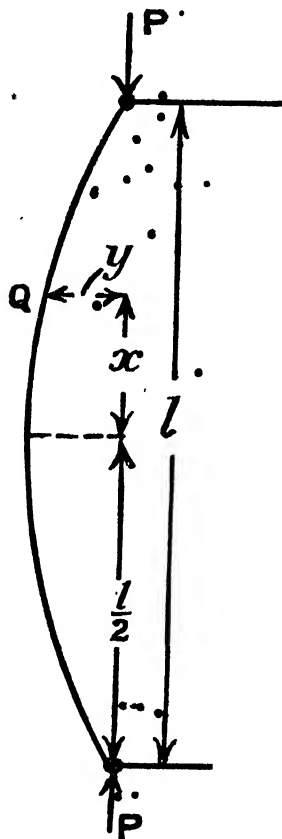


FIG. 89.

as $\sin mx$ will become negative when x is negative while $\cos mx$ is unchanged, A must be zero.

$$\therefore y = B \cos mx.$$

Also when $x = \pm \frac{l}{2}$, $y = 0$.

$$\therefore 0 = B \cos \frac{ml}{2} \quad \dots \dots \dots (3)$$

The smallest angle whose cosine is 0 is 90° or $\frac{\pi}{2}$.

$$\therefore \frac{ml}{2} = \frac{\pi}{2}$$

$$\text{i.e. } m = \frac{\pi}{l}$$

$$\text{or } m^2 = \frac{\pi^2}{l^2}$$

$$\text{but } m^2 = \frac{P}{EI}$$

$$\therefore \frac{P}{EI} = \frac{\pi^2}{l^2}$$

$$\text{or } P = \frac{\pi^2 EI}{l^2}$$

This is *Euler's formula*.

It gives us the load P which is theoretically required to maintain the column in an assumed deflected position, and it will be noted that this load is quite independent of the amount of the deflection and that if the deflection were doubled the load to keep it in such a deflected form would be the same. The load P therefore is called the *critical load upon the column* and with this load the column is in a state of unstable equilibrium.

If, therefore, the load upon the column is less than P and the column were given a slight deflection the column would straighten itself again, but if the load is ever so little in excess of P the deflection will go on increasing until the column fails. P thus may

be regarded as the load which will cause the column to fail.

Equation (3) would also be satisfied by

$$m \frac{l}{2} = \frac{3\pi}{2}$$

or $\frac{5\pi}{2}$. . . or any odd multiple of $\frac{\pi}{2}$,

but these solutions would be of no practical value because they would give larger values of P and the column would have buckled before such values were reached.

DIFFERENTIAL EQUATIONS IN GENERAL.—It is thought that the above simple examples will give an idea of the nature of differential equations. Further treatment is outside the scope of the present book and the general consideration of such equations forms one of the most difficult branches of advanced mathematics. The student who wishes to go further into the matter should consult Edwards' "Integral Calculus for Beginners," or Perry's "Calculus for Engineers". A more complete treatment will be found in Forsyth's "Differential Equations" [Macmillan].

EXERCISES 9.

1. Prove that the time of swing of a simple pendulum of length l which has a small swing is equal to

$$2\pi \sqrt{\frac{l}{g}}$$

[If θ is the angle of displacement,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta.]$$

2. A body is urged forward by a constant force A and there is a resistance proportional to its velocity.

Determine its equation of motion. [Proceed as follows :—

$$\text{Resultant force} = F = A - bv$$

$$\therefore W \cdot a = A - bv$$

$$a = \frac{Ag}{W} - \frac{bg}{W} \cdot v$$

$$= B - kv,$$

where B and k are constants.

$$\therefore a = \frac{dv}{dt} = B - kv.$$

Solve this for v and check your result by differentiation.]

3. Solve the equation $\frac{dy}{dx} + 2 \frac{y}{x} = x^2$.

[Multiply by x^2

$$x^2 \frac{dy}{dx} + 2xy = x^4$$

$$\text{i.e. } x^2 \frac{dy}{dx} + y \frac{dx^2}{dx} = x^4$$

$$\frac{d(x^2y)}{dx} = x^4.$$

Now integrate.]

4. Solve the equation $\frac{xdy}{dx} + x + y = 0$

$$[\text{i.e. } x \frac{dy}{dx} + y = -x]$$

$$\frac{d(xy)}{dx} = -x.$$

Now integrate.]

5. The equation of motion of two weights W and w connected by a string over a pulley is

$$(W + w) \frac{d^2s}{dt^2} = (W - w)g.$$

If $W = 3$ and $w = 2$ find the distance traversed after 1 second from starting.

ANSWERS TO EXERCISES.

9.

$$\begin{array}{ll}
 2. \quad v = Ce^{-kt} + \frac{B}{k} & 3. \quad x^2 y = \frac{x^5}{5} + C, \text{ i.e. } y \\
 = \frac{x^3}{5} + \frac{C}{x^2} & 4. \quad x^3 + 2xy = \text{constant.} \quad 5. \quad 3.2 \text{ ft.}
 \end{array}$$

CHAPTER X.

SOME FURTHER GEOMETRICAL APPLICATIONS AND SPECIAL CURVES.

§ 66. **Radius of Curvature.**—If QPS (Fig. 90) are three points on any curve and a circle be drawn through the points, the *circle of curvature* may be defined as the limiting position of this circle as the points Q and S both approach closer and closer to P.

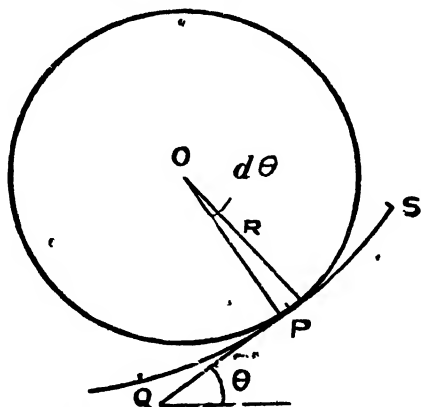


FIG. 90.

The centre O of this circle is called the *centre of curvature*, and since the radii O'O, S'Q' in Fig. 91 (which shows the limiting position of the points drawn to an enlarged scale), are at right angles

(normal) to the circle at O' and S' , and in the limit also normal to the curve, the centre of curvature may be defined as the intersection of two normals to the curve at points infinitely close to each other.

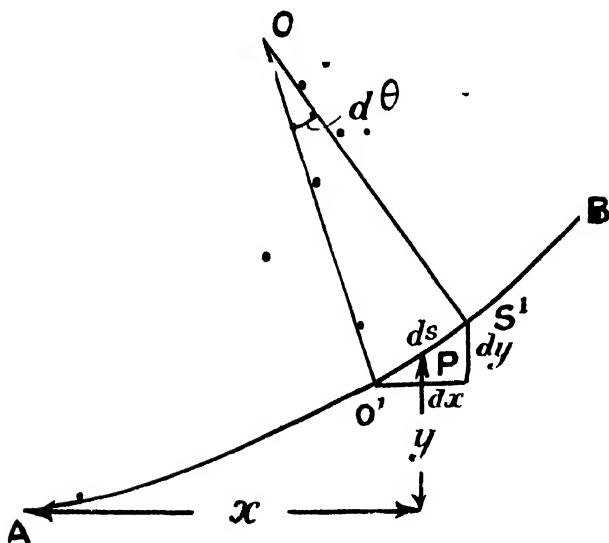


FIG. 91.

If $d\theta$ is the angle subtended at the centre by the element ds which is also an arc of the circle

$$ds = R d\theta$$

$$\text{or } R = \frac{ds}{d\theta} \quad (1)$$

The quantity $\frac{1}{R} \left(= \frac{d\theta}{ds} \right)$ is called the *curvature* of the curve at the point P.

To get equation (1) in terms of ordinary or

Cartesian co-ordinates, we put $\tan \theta = \frac{dy}{dx}$ (see § 4) and differentiating with regard to s

$$\begin{aligned} \therefore \frac{d(\tan \theta)}{ds} &= \frac{d}{ds} \left(\frac{dy}{dx} \right) \\ \text{i.e. } \frac{d \tan \theta}{d\theta} \cdot \frac{d\theta}{ds} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) \cdot \frac{dx}{ds} (*) \\ \text{i.e. } \sec^2 \theta \cdot \frac{d\theta}{ds} &= \frac{d^2y}{dx^2} \cdot \cos^2 \theta \\ \text{i.e. } \frac{1}{\cos^2 \theta} \cdot R &= \frac{d^2y}{dx^2} \cdot \cos^2 \theta \\ \text{i.e. } \frac{1}{R} &= \frac{d^2y}{dx^2} \cdot \cos^4 \theta \end{aligned} \quad (2)$$

$$\begin{aligned} \text{now } \cos \theta &= \frac{dx}{\sqrt{dx^2 + dy^2}} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \\ \therefore \cos^4 \theta &= \left(\frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right)^4 = \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{-2} \end{aligned}$$

putting this in equation (2) we get

$$\frac{1}{R} = \frac{d^2y}{dx^2} \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{-2} \quad (3)$$

We stated on p. 210 that when a curve such as the form of a slightly deflected beam is very flat, its curvature is given approximately by $\frac{1}{R} = \frac{d^2y}{dx^2}$; we can see from (3)* that this is the case because for a very flat curve the slope $\frac{dy}{dx}$ will be very small, and

* On each side here we have an example of $\frac{ds}{dx} \approx \frac{dx}{dy} \cdot \frac{dy}{dx}$

if we neglect it altogether we get the result $\frac{1}{R} = \frac{d^2y}{dx^2}$.

EXAMPLE OF A PARABOLA.—Take the case of the parabola $y = 8x^2$ and find the radius of curvature at the origin and for the value of $x = 2$.

$$\begin{aligned}\text{In this case } \frac{dy}{dx} &= \frac{d8x^2}{dx} = 16x, \\ \frac{d^2y}{dx^2} &= 16, \\ \therefore \frac{1}{R} &= \frac{16}{\{1 + 256x^2\}^{\frac{3}{2}}}, \\ R &= \frac{\{1 + 256x^2\}^{\frac{3}{2}}}{16}.\end{aligned}$$

At the origin where $x = 0$ this gives $R = \frac{1}{16}$;
at the point where $x = 2$

$$\begin{aligned}R &= \frac{(1 + 1024)^{\frac{3}{2}}}{16} \\ &= \frac{32^3}{16} \text{ nearly} \\ &= 2048.\end{aligned}$$

*We shall give other applications later in considering some special curves.

§ 67. The Length of Curved Outlines.—Referring to Fig. 91, we see that if we assume that the length of a very short piece ds of the curve is equal to the length of the chord $O'S$ we may write for the element of curve

$$\begin{aligned}ds^2 &= dx^2 + dy^2, \\ \text{i.e. } \left(\frac{ds}{dx}\right)^2 &= 1 + \left(\frac{dy}{dx}\right)^2, \\ \therefore \frac{ds}{dx} &= \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}, \\ \therefore ds &= \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx.\end{aligned}$$

If therefore s is the length of the curve from the point A to the point B

$$s = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \dots \quad (1)$$

or taking co-ordinates in the other direction

$$s = \int_A^B \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \quad \dots \quad (2)$$

We shall have an example of this in dealing with the catenary.

§ 68. **The Catenary and the Tractrix.**—The catenary is the curve in which a uniform cable hangs when supporting its own weight only. Its equation is of the form

$$y = \frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) \quad \dots \quad (1)$$

where a is a constant and it may also be written

$$y = a \cosh \frac{x}{a} \quad \dots \quad (2)$$

* It is usual to write the expression $\frac{1}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right)$ as $\cosh \frac{x}{a}$ and the expression $\frac{1}{2} \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right)$ as $\sinh \frac{x}{a}$, these being contracted forms for *hyperbolic cosine* and *hyperbolic sine* respectively, because there is an analogy between these functions with reference to a rectangular hyperbola and ordinary trigonometrical functions with reference to a circle. It will be noted that $\cosh^2 \left(\frac{x}{a} \right) - \sinh^2 \left(\frac{x}{a} \right) = 1$. Values of these functions are tabulated on p. 239.

It follows that

$$\frac{d \cosh x}{dx} = \sinh x$$

$$\text{and } \frac{d \sinh x}{dx} = \cosh x.$$

This should be tested by the student.

The axis of x is called the *directrix* in this case.

This curve has many interesting properties.

Take first the length of the curve from A to P (Fig. 92).

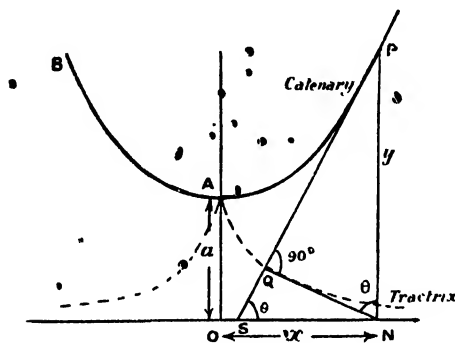


FIG. 92.

$$\begin{aligned}\frac{dy}{dx} &= \frac{a}{2} \left(\frac{1}{a} e^{\frac{x}{a}} - \frac{1}{a} e^{-\frac{x}{a}} \right) \\ &= \frac{1}{2} \left(e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right) \quad \dots \quad (3)\end{aligned}$$

$$\begin{aligned}\left(\frac{dy}{dx} \right)^2 &= \frac{1}{4} \left(e^{\frac{2x}{a}} + e^{-\frac{2x}{a}} - 2e^{\frac{x}{a}} e^{-\frac{x}{a}} \right) \\ &= \frac{1}{4} \left(e^{\frac{2x}{a}} + e^{-\frac{2x}{a}} \right) - \frac{1}{2}\end{aligned}$$

$$\therefore 1 + \left(\frac{dy}{dx} \right)^2 = \frac{1}{4} \left(e^{\frac{2x}{a}} + e^{-\frac{2x}{a}} \right) + \frac{1}{2}$$

$$= \left\{ \frac{1}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) \right\}^2$$

$$= \frac{y^2}{a^2} \quad ; \quad \dots \quad ; \quad (4)$$

The length of the curve from A to P

$$= s = \int_a^x \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \cdot dx,$$

$$\begin{aligned}
 &= \int_a^x y \, dx \\
 &= \int_a^x \frac{1}{2} (e^x + e^{-x}) \, dx \\
 &= \frac{a}{2} \left[e^x - e^{-x} \right]_a^x \quad \dots \dots \dots (5)
 \end{aligned}$$

The expression vanishes at the lower limit, for $e^0 = 1$.

$$\therefore s = \frac{a}{2} (e^x - e^{-x}) \quad \dots \dots \dots (6)$$

$$= a \sinh \left(\frac{x}{a} \right) \quad \dots \dots \dots (7)$$

Going back to equation (3) we see that

$$s = a \cdot \frac{dy}{dx} \quad \dots \dots \dots (8)$$

Referring to the figure, we note that

$$\frac{dy}{dx} = \tan \theta = \frac{PN}{SN}$$

Now $1 + \tan^2 \theta = \sec^2 \theta$

$$\therefore \sec^2 \theta = 1 + \left(\frac{dy}{dx} \right)^2 = \frac{y^2}{a^2} \text{ [from (4)]}$$

$$\therefore \sec \theta = \frac{y}{a} \quad \dots \dots \dots (9)$$

Now draw NQ perpendicular to the tangent PS.

Then $\angle NSP = \theta = \frac{\pi}{2} - \angle SPN = \angle PNQ$,

$$\text{and } \sec \theta = \frac{PN}{QN} = \frac{y}{QN} = \frac{y}{a} \text{ [from (9)]}.$$

$$\therefore QN = a \quad \dots \dots \dots (10)$$

$$\text{Also } \frac{PQ}{QN} = \tan \theta = \frac{dy}{dx}$$

$$\therefore PQ = a \frac{dy}{dx} = s \text{ [from (8)]}.$$

If, therefore, a string PQ held taut were wrapped round the catenary, Q would describe a curved path shown in dotted lines, and the point Q would come to the vertex A because $PQ = s$; this path is called the *tractrix*.

When two curves are so related that the second is the locus of the end of a string as it is unwound from the first, the second curve is called an *involute* of the first, so that the tractrix is an involute of the catenary.

We notice also that QN is tangential to the tractrix because it is at right angles to PQ and the instantaneous path of Q is perpendicular to PQ, and we have proved that its length is equal to a which is constant. The tractrix therefore has the property that its tangent is of constant length [see p. 179].

The tractrix gets its name from the fact that it is the curve moved through by a body placed at A and connected by a string of length a which is moved along ON so slowly that the body acquires no momentum — or the body may be supposed “rough”.

PROOF THAT A CABLE HANGS IN THE FORM OF A CATENARY.—Consider the equilibrium of a portion PA (Fig. 93) of the cable of length s ; there are three forces acting on it, viz. a horizontal tension T_0 at A, a tangential tension T at P, and a weight ws , s being the length of the cable from A to P, and w being the weight per unit length of the cable, the weight being assumed constant.

Then clearly from the triangle of forces $a b c$

$$ws = T_0 \tan \theta = T_0 \frac{dy}{dx}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{ws}{T_0} \\ \text{i.e. } s &= \frac{T_0 dy}{w \frac{dy}{dx}} \\ &= a \cdot \frac{dy}{dx} \end{aligned} \quad (1)$$

a being a constant because T_0 and w are both constant.

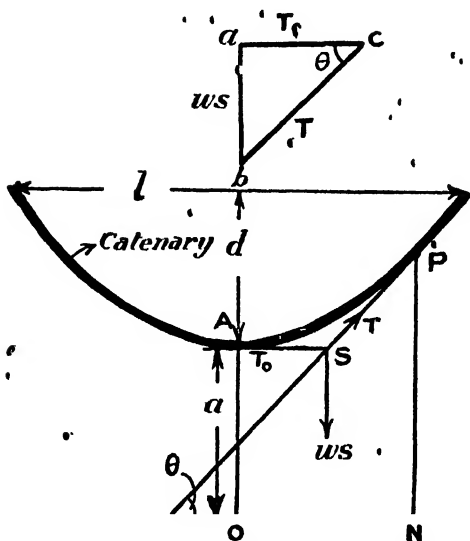


FIG. 93.

But we have already shown in equation (8) that this relation holds for the catenary, so that the catenary is a curve fulfilling the necessary conditions of stability of the cable.

We also see that since $a = \frac{T_0}{w}$, the horizontal tension T_0 at A is $w \cdot AO$.

Now consider the tension T at the point P .

It follows from the triangle of forces that

$$\frac{T}{T_0} = \sec \theta,$$

but from Fig. 92, $\sec \theta = \frac{PN}{QN} = \frac{y}{a}$

$$\begin{aligned} \therefore T &= \frac{T_0 y}{a} \\ &= w a \cdot \frac{y}{a} \\ &= wy. \end{aligned}$$

The tension therefore at any point of the cable is equal to the weight per unit length of the cable multiplied by the distance from the point to the directrix of the catenary.

Engineers usually regard the catenary as a troublesome curve to deal with in cable problems because if the span and dip are given the calculation of a , which is necessary before the curve can be drawn, is not obvious.

With the aid of tables of hyperbolic functions in the appendix, however, we can find a as follows:—

If l is the span of the cable and d is the dip,

$$y = a + d \text{ when } x = \frac{l}{2}$$

$$\therefore a + d = a \cosh \frac{l}{2a}$$

$$\therefore 1 + \frac{d}{a} = \cosh \frac{l}{2a}$$

$$\therefore \text{If } \frac{l}{2a} = z$$

$$1 + \frac{2d}{l} z = \cosh z.$$

z can then be found as follows by plotting from the tables: a catenary $y = \cosh z$ (Fig. 94). Take for

instance $\frac{l}{a} = 10$.

$$\text{Then } \frac{2d}{l} = \frac{2}{10} = .2$$

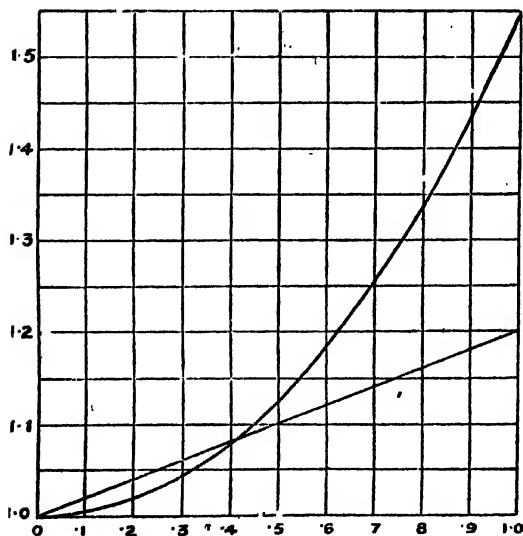


FIG. 94.

\therefore draw the straight line $y = 1 + .2z$ starting from the point $y = 1$ as shown.

The straight line cuts the catenary at $z = .42$

$$\therefore \frac{l}{2a} = .42 \text{ or } a = \frac{l}{.84} = 1.19l.$$

If we are given the length $2s$ and span l and not the dip we use the relation

$$\begin{aligned} s &= a \sinh \frac{x}{a} \\ &= a \sinh \frac{l}{a} \\ \text{i.e. } \frac{s}{l} &= \frac{a}{l} \sinh \frac{l}{a} \\ \text{or if } \frac{s}{l} &= b \text{ and } \frac{l}{a} = z \\ bz &= \sinh z \end{aligned}$$

b is given and so by plotting a curve of $\sinh z$ and drawing upon it the line representing bz we can find the solution in a similar manner.

§ 69. **Envelopes, Evolutes, and Involute.**—The *envelope* of a series of curves of the same family, i.e. those obeying the same kind of law, may be defined as the locus of the points of intersection of the curves when such curves are infinitely close; it may be regarded as the boundary curve within or without which all the particular curves of the family lie.

A familiar example of this arises in the case of the geometrical construction shown in Fig. 95 for the parabola. The sides of the triangle are divided up into an equal number of parts and joined across as shown; the greater the number of parts, the nearer will the path of the successive intersection approach to the parabola which is thus the envelope of this particular family of straight lines.

There is also an example of envelopes in the case of the bending moment and shear diagrams for a rolling

load upon a beam which will be found described in text-books upon the theory of structures.

The *evolute* of a curve is the locus or path of its centre of curvature. The circle is the curve whose centre of curvature is fixed so that the evolute of a circle is a fixed point, viz. the centre of the circle.

The centre of curvature may be regarded as the intersection of successive normals since the centre of

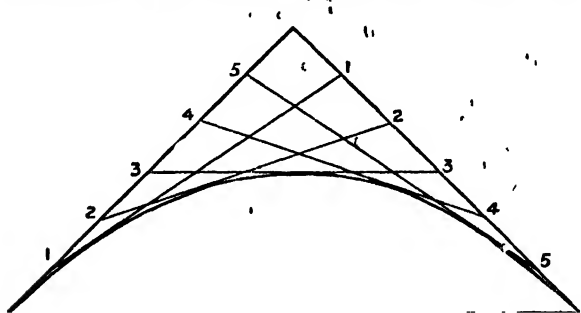


FIG. 95.

curvature is the ultimate point of intersection of two close normals, the evolute is the envelope of the normals to the given curve. The normals to the original curve are therefore the tangents to the evolute.

An *involute* of any given curve is the locus of points obtained by marking along the tangent to the curve at any point a length equal to the length of the curve from some fixed point upon it. We have already had an example of this in the case of the tractrix. If a curve A be the evolute of a curve B, B will be an involute of A.

A curve has an infinite number of involutes, one corresponding to each fixed point on the curve.

As a more practical definition of an involute, we may imagine a string to be wrapped round the given curve and a pencil fixed to a point along the string; if the string then be unwound while being kept taut, the pencil will draw an involute of the curve.

The original curve is the evolute of all its involutes.

THE CIRCULAR INVOLUTE.—The involute which has the greatest practical application is the circular involute which is used for the shape of gear wheel teeth. If we start unwinding at the point A (Fig. 96) we get the curve AQx, so that if SQ is the tangent to the circle at a point S, the length SQ will be equal to the arc AS.

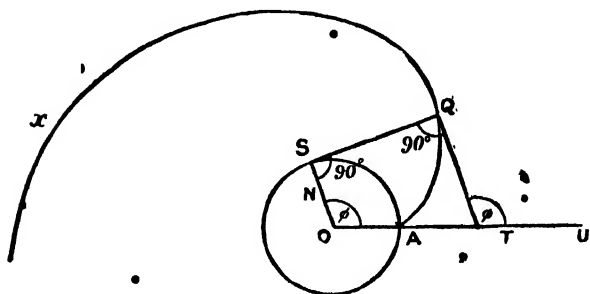


FIG. 96.

Since we take the starting-point A at any point on the curve there must be, as we have already stated, an infinite number of involutes to every curve, and in the case of the circle all the involutes will have the same shape.

At any point Q of the involute the centre of curvature is the corresponding point S on the circle because S is the instantaneous centre of movement of the string. We have already shown on p. 216, that

so that their common tangent SS' passes through the pitch point P .

Let the outlines TT_1 be involutes of these base circles; then it is clear from the figure that SS' is the common normal to the two curves at the point of contact J of the two teeth, and SS' passes through the pitch point P , so that the involute fulfils the necessary condition for gear teeth.

§ 70. **Cycloids.**—The cycloids are a series of curves which also arise in problems of toothed gearing.

The cycloid is the curve traced out by a point upon a circle which rolls without slip upon a straight line.

Referring to Fig. 98, if r is the radius of the

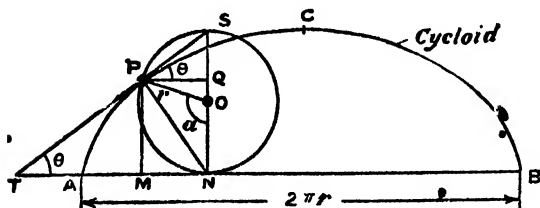


FIG. 98.

rolling circle, the point P will describe the cycloid ACB , the base AB of which will be equal to $2\pi r$, the circumference of the circle.

Take any position of the rolling circle.

Then the arc PN must be equal to the length AN if no slip occurs.

$$AN = ra$$

$$MP = y = ON + OQ = r + r \cos (180^\circ - a)$$

$$y = r(1 - \cos a)$$

$$MN \perp PQ = r \sin (180 - a) = r \sin a$$

$$\therefore AM = x = AN - MN$$

$$x = r (a - \sin a) \quad (2)$$

Let PT be the tangent at the point P, then

$$\tan \theta = \frac{dy}{dx}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \div \frac{dx}{du}$$

$$\text{Now } \frac{dy}{du} = \frac{d \{ r(1 - \cos a) \}}{du}$$

$$= r (0 + \sin a) = r \sin a \quad (3)$$

$$\frac{dx}{du} = \frac{d \{ r(a - \sin a) \}}{da}$$

$$= r (1 - \cos a) \quad (4)$$

$$\tan \theta = \frac{r \sin a}{r (1 - \cos a)}$$

$$= \frac{\sin a}{1 - \cos a} = \frac{2 \sin \frac{a}{2} \cos \frac{a}{2}}{2 \sin^2 \frac{a}{2}}$$

$$= \frac{\cos \frac{a}{2}}{\sin \frac{a}{2}} = \cot \frac{a}{2} \quad (5)$$

$$\text{but } \tan \theta = \cot (90 - \theta) = \cot \angle PSQ$$

$$\therefore \angle PSQ = \frac{a}{2}$$

This means that S is on the diameter of the circle.

This can be seen more simply. Since the circle is rolling along AB, its point of contact N is the instantaneous centre about which the circle is rotating so that PN is the normal at P and PS the tangent to the curve, and since the angle in a semi-circle is a right angle NS must be a diameter of the circle.

LENGTH OF ARC OF CYCLOID.—We have shown on p. 219 that

$$\begin{aligned}
 ds^2 &= dx^2 + dy^2 \\
 \therefore \left(\frac{ds}{du}\right)^2 &= \left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 \\
 &= r^2 (1 - \cos u)^2 + r^2 \sin^2 u \\
 &= r^2 (1 - 2 \cos u + \cos^2 u + \sin^2 u) \\
 &= r^2 (2 - 2 \cos u) \text{ because } \sin^2 u + \cos^2 u = 1 \\
 &= 2r^2 (1 - \cos u) \\
 &= 4r^2 \sin^2 \frac{u}{2}
 \end{aligned}$$

$$\therefore \frac{ds}{du} = 2r \sin \frac{u}{2} \quad (6)$$

$$\therefore \text{length of arc AP} = s = \int_0^u 2r \sin \frac{u}{2} du$$

$$= -2r \left[2 \cos \frac{u}{2} \right]_0^u$$

$$= -2r \left(2 \cos \frac{u}{2} - 2 \right)$$

$$= 4r \left(1 - \cos \frac{u}{2} \right) \quad (7)$$

$$\begin{aligned}
 \therefore \text{Length of whole curve AB for which } u &= 360^\circ \\
 &= 4r \{1 - (-1)\} = 8r \quad (8)
 \end{aligned}$$

RADIUS OF CURVATURE OF CYCLOID.—

$$R = \frac{ds}{d\theta}$$

$$= \frac{ds}{du} \cdot \frac{du}{d\theta} = 2r \sin \frac{u}{2} \div \frac{d\theta}{du} [\text{from (6)}].$$

Now we have shown that $\theta = 90 - \frac{u}{2}$

$$\text{and } \frac{d\theta}{du} = -\frac{1}{2}$$

$$\begin{aligned}
 \therefore R &= -2r \cos \theta \div \frac{1}{2} \\
 &= -4r \cos \theta \\
 &= -2 \cdot 2r \cos \text{PNS} \\
 &= -2PN \quad \quad \quad (9)
 \end{aligned}$$

The sign may be adjusted by convention; note that either sign may be attached to formula (6).

THE TROCHOID.—The path generated by some point fixed relatively to the circle but not situated on the circumference is called a trochoid. If the point is outside the circle, the trochoid has loops, as shown in Fig 99

EPICYCLOID AND HYPOCYCLOID.—When a circle

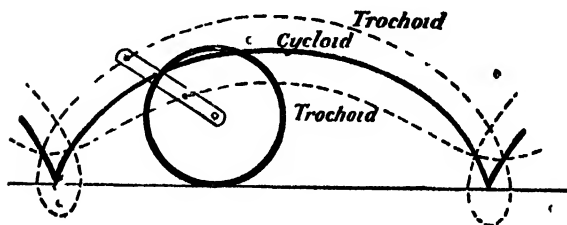


FIG 99.

rolls outside another circle without slipping, the resulting curve is called an epicycloid, while if it rolls inside, it is called a hypocycloid. The curves are shown on Fig. 100.

If the radius R of the main circle is an exact multiple (2, 3, 4, etc.) of the radius r of the rolling circle, the curves will repeat as the rolling circle continues to roll more than once round the main circle. If the rolling circle has a radius equal to half that of the main circle, the hypocycloid becomes the diameter. This can be proved from Fig. 101 in which A

represents the starting-point and OP represents any given position of the diameter of the rolling circle.

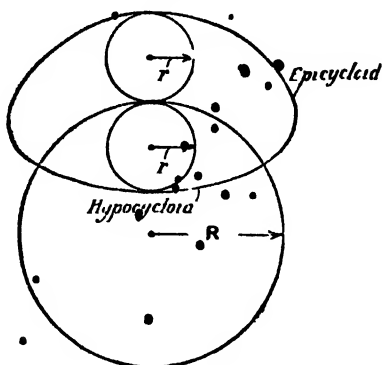


FIG. 100.

Then if $\angle AOP = \theta$, $\angle QSP = 2\theta$ because the

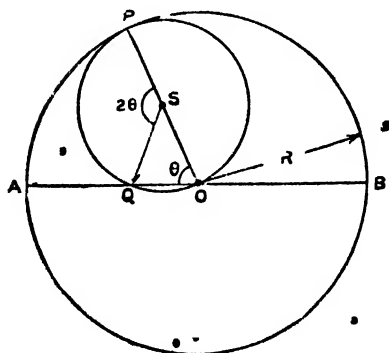


FIG 101.

angle at the centre of a circle is twice that at the circumference

Now arc $AP = R\theta$

and arc $QP = r \cdot 2\theta$, so since $R = 2r$

arc $AP = \text{arc } PQ$

$\therefore Q$ which is on the diameter AB is also on the hypocycloid.

APPLICATION TO TOOTHED GEARING.—Referring to Fig. 102, let the pitch circles APB and CPD touch at P and let ab and cd be portions of hypocycloidal and epicycloidal tooth outlines touching at Q , the common rolling circle being QPR .

ab is the curve for rolling upon APB and dc is that for rolling upon CPD . At the instant, the circle

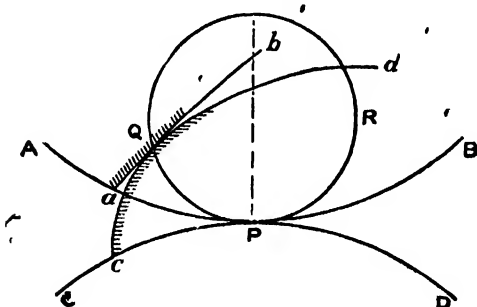


FIG. 102.

QPR is rolling outside CPD and inside APB and the direction of each curve must be normal to the line QP because this circle is rolling about P as an instantaneous centre.

The two curves therefore fulfil the necessary condition that the common normal shall pass through the pitch-point P .

These outlines determine only the root of one tooth and the point of the other; the other portions of each teeth can be obtained similarly but need not

necessarily have the same radius of rolling circle as the first portions.

Cycloidal teeth are now seldom used in practice.

EXERCISES 10.

1. Find the radius of curvature at the point $x = 1$ of the curve $y^2 = x^3$.

2. Find the radius of curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3. Find the radius of curvature at the point $x = 2$ of the hyperbola $xy = 4$.

4. Find the radius of curvature of the catenary $y = a \cosh \frac{x}{a}$.

5. Find the radius of curvature of the cubic parabola $ay^2 = x^3$.

6. Find an expression for the length of the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

7. Find the length of the curve $9ay^2 = 4x^3$ starting from the point $x = 0$.

8. Draw a curve $y = \sinh x$ and find the value of a for a cable whose length is twice the span. (See p. 226.)

9. Find the length of the parabola $y^2 = 4ax$ from the vertex.

10. If R is the radius of the fixed circle and r that of the rolling circle, show that the co-ordinates of a point on the epicycloid may be written:—

$$x = (R + r) \cos a + b \cos \left(\frac{R + r}{r} a \right)$$

$$y = (R + r) \sin a + b \sin \left(\frac{R + r}{r} a \right)$$

a being the angle turned through by the point of contact.

ANSWERS TO EXERCISES.

EXERCISES 10.

$$1. \frac{13^{\frac{3}{2}}}{6} = 7.81. \quad 2. -b^4x^2. \quad 3. 2\sqrt{2}. \quad 4. \frac{y^2}{a}$$

$$5. \frac{a^2}{bx} \left(1 + \frac{9x^4}{a^4}\right)^{\frac{3}{2}}. \quad 6. \frac{3a^{\frac{1}{2}}x^{\frac{3}{2}}}{2}$$

$$7. \frac{2a}{3} \left\{ \left(1 + \frac{x}{a}\right)^{\frac{3}{2}} - 1 \right\}. \quad 8. a = .455l.$$

$$9. s = \sqrt{ax} + x^2 + a \log \left(\frac{\sqrt{x^2 + a} + \sqrt{a + x}}{\sqrt{a}} \right).$$

EXPONENTIAL AND HYPERBOLIC FUNCTIONS.

| x | e^x | e^{-x} | $\cosh x$ | $\sinh x$ |
|-----|--------|----------|-----------|-----------|
| 0 | 1.000 | 1.000 | 1.000 | 0 |
| .1 | 1.105 | .905 | 1.005 | .100 |
| .2 | 1.221 | .819 | 1.020 | .201 |
| .3 | 1.350 | .741 | 1.045 | .305 |
| .4 | 1.492 | .670 | 1.081 | .411 |
| .5 | 1.649 | .607 | 1.128 | .521 |
| .6 | 1.822 | .549 | 1.185 | .637 |
| .7 | 2.014 | .497 | 1.255 | .759 |
| .8 | 2.226 | .449 | 1.337 | .888 |
| .9 | 2.460 | .407 | 1.433 | 1.027 |
| 1.0 | 2.718 | .368 | 1.543 | 1.175 |
| 1.1 | 3.004 | .333 | 1.669 | 1.336 |
| 1.2 | 3.320 | .301 | 1.811 | 1.509 |
| 1.3 | 3.669 | .273 | 1.971 | 1.698 |
| 1.4 | 4.055 | .247 | 2.151 | 1.904 |
| 1.5 | 4.482 | .223 | 2.352 | 2.129 |
| 1.6 | 4.953 | .202 | 2.577 | 2.366 |
| 1.7 | 5.474 | .183 | 2.828 | 2.616 |
| 1.8 | 6.050 | .165 | 3.107 | 2.942 |
| 1.9 | 6.686 | .150 | 3.418 | 3.268 |
| 2.0 | 7.389 | .135 | 3.762 | 3.627 |
| 2.1 | 8.166 | .122 | 4.144 | 4.022 |
| 2.2 | 9.025 | .111 | 4.568 | 4.457 |
| 2.3 | 9.974 | .100 | 5.037 | 4.937 |
| 2.4 | 11.023 | .091 | 5.557 | 5.466 |
| 2.5 | 12.182 | .082 | 6.132 | 6.050 |

HYPERBOLIC LOGARITHMS OF NUMBERS FROM 1 TO 50.

| No. | Log. | No. | Log. | No. | Log. | No. | Log. |
|------|-------|------|-------|------|-------|------|-------|
| 1.00 | .0000 | 1.40 | .3365 | 1.80 | .5878 | 2.20 | .7885 |
| 1.01 | .0099 | 1.41 | .3436 | 1.81 | .5933 | 2.21 | .7930 |
| 1.02 | .0198 | 1.42 | .3507 | 1.82 | .5988 | 2.22 | .7975 |
| 1.03 | .0296 | 1.43 | .3577 | 1.83 | .6043 | 2.23 | .8020 |
| 1.04 | .0392 | 1.44 | .3646 | 1.84 | .6098 | 2.24 | .8065 |
| 1.05 | .0488 | 1.45 | .3716 | 1.85 | .6152 | 2.25 | .8109 |
| 1.06 | .0583 | 1.46 | .3784 | 1.86 | .6206 | 2.26 | .8154 |
| 1.07 | .0677 | 1.47 | .3853 | 1.87 | .6259 | 2.27 | .8198 |
| 1.08 | .0770 | 1.48 | .3920 | 1.88 | .6313 | 2.28 | .8242 |
| 1.09 | .0862 | 1.49 | .3988 | 1.89 | .6366 | 2.29 | .8286 |
| 1.10 | .0953 | 1.50 | .4055 | 1.90 | .6419 | 2.30 | .8329 |
| 1.11 | .1044 | 1.51 | .4121 | 1.91 | .6471 | 2.31 | .8372 |
| 1.12 | .1133 | 1.52 | .4187 | 1.92 | .6523 | 2.32 | .8416 |
| 1.13 | .1222 | 1.53 | .4253 | 1.93 | .6575 | 2.33 | .8458 |
| 1.14 | .1310 | 1.54 | .4318 | 1.94 | .6627 | 2.34 | .8502 |
| 1.15 | .1398 | 1.55 | .4383 | 1.95 | .6678 | 2.35 | .8544 |
| 1.16 | .1484 | 1.56 | .4447 | 1.96 | .6729 | 2.36 | .8587 |
| 1.17 | .1570 | 1.57 | .4511 | 1.97 | .6780 | 2.37 | .8629 |
| 1.18 | .1655 | 1.58 | .4574 | 1.98 | .6831 | 2.38 | .8671 |
| 1.19 | .1740 | 1.59 | .4637 | 1.99 | .6881 | 2.39 | .8713 |
| 1.20 | .1823 | 1.60 | .4700 | 2.00 | .6931 | 2.40 | .8755 |
| 1.21 | .1906 | 1.61 | .4762 | 2.01 | .6981 | 2.41 | .8796 |
| 1.22 | .1988 | 1.62 | .4824 | 2.02 | .7031 | 2.42 | .8838 |
| 1.23 | .2070 | 1.63 | .4886 | 2.03 | .7080 | 2.43 | .8879 |
| 1.24 | .2151 | 1.64 | .4947 | 2.04 | .7129 | 2.44 | .8920 |
| 1.25 | .2231 | 1.65 | .5008 | 2.05 | .7178 | 2.45 | .8961 |
| 1.26 | .2311 | 1.66 | .5068 | 2.06 | .7227 | 2.46 | .9002 |
| 1.27 | .2390 | 1.67 | .5128 | 2.07 | .7275 | 2.47 | .9042 |
| 1.28 | .2469 | 1.68 | .5188 | 2.08 | .7324 | 2.48 | .9083 |
| 1.29 | .2546 | 1.69 | .5247 | 2.09 | .7372 | 2.49 | .9123 |
| 1.30 | .2624 | 1.70 | .5306 | 2.10 | .7419 | 2.50 | .9163 |
| 1.31 | .2700 | 1.71 | .5365 | 2.11 | .7467 | 2.51 | .9203 |
| 1.32 | .2776 | 1.72 | .5423 | 2.12 | .7514 | 2.52 | .9243 |
| 1.33 | .2852 | 1.73 | .5481 | 2.13 | .7561 | 2.53 | .9282 |
| 1.34 | .2927 | 1.74 | .5539 | 2.14 | .7608 | 2.54 | .9322 |
| 1.35 | .3001 | 1.75 | .5596 | 2.15 | .7655 | 2.55 | .9361 |
| 1.36 | .3075 | 1.76 | .5653 | 2.16 | .7701 | 2.56 | .9400 |
| 1.37 | .3148 | 1.77 | .5710 | 2.17 | .7747 | 2.57 | .9439 |
| 1.38 | .3221 | 1.78 | .5766 | 2.18 | .7793 | 2.58 | .9478 |
| 1.39 | .3293 | 1.79 | .5822 | 2.19 | .7839 | 2.59 | .9517 |

| No. | Log. | No. | Log. | No. | Log. | No. | Log. |
|------|--------|------|--------|------|--------|------|--------|
| 2-60 | ·9555 | 3-94 | 1·1119 | 3-48 | 1·2470 | 3-92 | 1·8661 |
| 2-61 | ·9594 | 3-95 | 1·1151 | 3-49 | 1·2499 | 3-93 | 1·8686 |
| 2-62 | ·9632 | 3-96 | 1·1184 | 3-50 | 1·2528 | 3-94 | 1·8712 |
| 2-63 | ·9670 | 3-97 | 1·1217 | 3-51 | 1·2556 | 3-95 | 1·8737 |
| 2-64 | ·9708 | 3-98 | 1·1249 | 3-52 | 1·2585 | 3-96 | 1·8762 |
| 2-65 | ·9746 | 3-99 | 1·1282 | 3-53 | 1·2613 | 3-97 | 1·8788 |
| 2-66 | ·9783 | 3-10 | 1·1314 | 3-54 | 1·2641 | 3-98 | 1·8813 |
| 2-67 | ·9821 | 3-11 | 1·1346 | 3-55 | 1·2669 | 3-99 | 1·8838 |
| 2-68 | ·9858 | 3-12 | 1·1378 | 3-56 | 1·2698 | 4-00 | 1·8863 |
| 2-69 | ·9895 | 3-13 | 1·1410 | 3-57 | 1·2726 | 4-01 | 1·8888 |
| 2-70 | ·9933 | 3-14 | 1·1442 | 3-58 | 1·2754 | 4-02 | 1·8913 |
| 2-71 | ·9969 | 3-15 | 1·1474 | 3-59 | 1·2782 | 4-03 | 1·8938 |
| 2-72 | 1·0006 | 3-16 | 1·1506 | 3-60 | 1·2809 | 4-04 | 1·8962 |
| 2-73 | 1·0043 | 3-17 | 1·1537 | 3-61 | 1·2837 | 4-05 | 1·8987 |
| 2-74 | 1·0080 | 3-18 | 1·1569 | 3-62 | 1·2865 | 4-06 | 1·4012 |
| 2-75 | 1·0116 | 3-19 | 1·1600 | 3-63 | 1·2892 | 4-07 | 1·4036 |
| 2-76 | 1·0152 | 3-20 | 1·1632 | 3-64 | 1·2920 | 4-08 | 1·4061 |
| 2-77 | 1·0188 | 3-21 | 1·1663 | 3-65 | 1·2947 | 4-09 | 1·4085 |
| 2-78 | 1·0225 | 3-22 | 1·1694 | 3-66 | 1·2975 | 4-10 | 1·4110 |
| 2-79 | 1·0260 | 3-23 | 1·1725 | 3-67 | 1·3002 | 4-11 | 1·4134 |
| 2-80 | 1·0296 | 3-24 | 1·1756 | 3-68 | 1·3029 | 4-12 | 1·4159 |
| 2-81 | 1·0332 | 3-25 | 1·1787 | 3-69 | 1·3056 | 4-13 | 1·4183 |
| 2-82 | 1·0367 | 3-26 | 1·1817 | 3-70 | 1·3083 | 4-14 | 1·4207 |
| 2-83 | 1·0403 | 3-27 | 1·1848 | 3-71 | 1·3110 | 4-15 | 1·4231 |
| 2-84 | 1·0438 | 3-28 | 1·1878 | 3-72 | 1·3137 | 4-16 | 1·4255 |
| 2-85 | 1·0473 | 3-29 | 1·1909 | 3-73 | 1·3164 | 4-17 | 1·4279 |
| 2-86 | 1·0508 | 3-30 | 1·1939 | 3-74 | 1·3191 | 4-18 | 1·4303 |
| 2-87 | 1·0543 | 3-31 | 1·1969 | 3-75 | 1·3218 | 4-19 | 1·4327 |
| 2-88 | 1·0578 | 3-32 | 1·1999 | 3-76 | 1·3244 | 4-20 | 1·4351 |
| 2-89 | 1·0613 | 3-33 | 1·2030 | 3-77 | 1·3271 | 4-21 | 1·4375 |
| 2-90 | 1·0647 | 3-34 | 1·2060 | 3-78 | 1·3297 | 4-22 | 1·4398 |
| 2-91 | 1·0682 | 3-35 | 1·2090 | 3-79 | 1·3324 | 4-23 | 1·4422 |
| 2-92 | 1·0716 | 3-36 | 1·2119 | 3-80 | 1·3350 | 4-24 | 1·4446 |
| 2-93 | 1·0750 | 3-37 | 1·2149 | 3-81 | 1·3376 | 4-25 | 1·4469 |
| 2-94 | 1·0784 | 3-38 | 1·2179 | 3-82 | 1·3403 | 4-26 | 1·4493 |
| 2-95 | 1·0818 | 3-39 | 1·2208 | 3-83 | 1·3429 | 4-27 | 1·4516 |
| 2-96 | 1·0852 | 3-40 | 1·2238 | 3-84 | 1·3455 | 4-28 | 1·4540 |
| 2-97 | 1·0886 | 3-41 | 1·2267 | 3-85 | 1·3481 | 4-29 | 1·4563 |
| 2-98 | 1·0919 | 3-42 | 1·2296 | 3-86 | 1·3507 | 4-30 | 1·4586 |
| 2-99 | 1·0953 | 3-43 | 1·2326 | 3-87 | 1·3533 | 4-31 | 1·4609 |
| 3-00 | 1·0986 | 3-44 | 1·2355 | 3-88 | 1·3558 | 4-32 | 1·4633 |
| 3-01 | 1·1019 | 3-45 | 1·2384 | 3-89 | 1·3584 | 4-33 | 1·4656 |
| 3-02 | 1·1053 | 3-46 | 1·2413 | 3-90 | 1·3610 | 4-34 | 1·4679 |
| 3-03 | 1·1086 | 3-47 | 1·2442 | 3-91 | 1·3635 | 4-35 | 1·4702 |

| No. | Log. | No. | Log. | No. | Log. | No. | Log. |
|------|--------|------|--------|------|--------|------|--------|
| 4.36 | 1.4725 | 4.80 | 1.5686 | 5.24 | 1.6563 | 5.68 | 1.7370 |
| 4.37 | 1.4748 | 4.81 | 1.5707 | 5.25 | 1.6582 | 5.69 | 1.7387 |
| 4.38 | 1.4770 | 4.82 | 1.5728 | 5.26 | 1.6601 | 5.70 | 1.7405 |
| 4.39 | 1.4793 | 4.83 | 1.5748 | 5.27 | 1.6620 | 5.71 | 1.7422 |
| 4.40 | 1.4816 | 4.84 | 1.5769 | 5.28 | 1.6639 | 5.72 | 1.7440 |
| 4.41 | 1.4839 | 4.85 | 1.5790 | 5.29 | 1.6658 | 5.73 | 1.7457 |
| 4.42 | 1.4861 | 4.86 | 1.5810 | 5.30 | 1.6677 | 5.74 | 1.7475 |
| 4.43 | 1.4884 | 4.87 | 1.5831 | 5.31 | 1.6696 | 5.75 | 1.7492 |
| 4.44 | 1.4907 | 4.88 | 1.5851 | 5.32 | 1.6715 | 5.76 | 1.7509 |
| 4.45 | 1.4929 | 4.89 | 1.5872 | 5.33 | 1.6734 | 5.77 | 1.7527 |
| 4.46 | 1.4951 | 4.90 | 1.5892 | 5.34 | 1.6752 | 5.78 | 1.7544 |
| 4.47 | 1.4974 | 4.91 | 1.5913 | 5.35 | 1.6771 | 5.79 | 1.7561 |
| 4.48 | 1.4996 | 4.92 | 1.5933 | 5.36 | 1.6790 | 5.80 | 1.7579 |
| 4.49 | 1.5019 | 4.93 | 1.5953 | 5.37 | 1.6808 | 5.81 | 1.7596 |
| 4.50 | 1.5041 | 4.94 | 1.5974 | 5.38 | 1.6827 | 5.82 | 1.7613 |
| 4.51 | 1.5063 | 4.95 | 1.5994 | 5.39 | 1.6845 | 5.83 | 1.7630 |
| 4.52 | 1.5085 | 4.96 | 1.6014 | 5.40 | 1.6864 | 5.84 | 1.7647 |
| 4.53 | 1.5107 | 4.97 | 1.6034 | 5.41 | 1.6882 | 5.85 | 1.7664 |
| 4.54 | 1.5129 | 4.98 | 1.6054 | 5.42 | 1.6901 | 5.86 | 1.7681 |
| 4.55 | 1.5151 | 4.99 | 1.6074 | 5.43 | 1.6919 | 5.87 | 1.7699 |
| 4.56 | 1.5173 | 5.00 | 1.6094 | 5.44 | 1.6938 | 5.88 | 1.7716 |
| 4.57 | 1.5195 | 5.01 | 1.6114 | 5.45 | 1.6956 | 5.89 | 1.7733 |
| 4.58 | 1.5217 | 5.02 | 1.6134 | 5.46 | 1.6974 | 5.90 | 1.7750 |
| 4.59 | 1.5239 | 5.03 | 1.6154 | 5.47 | 1.6993 | 5.91 | 1.7766 |
| 4.60 | 1.5261 | 5.04 | 1.6174 | 5.48 | 1.7011 | 5.92 | 1.7783 |
| 4.61 | 1.5282 | 5.05 | 1.6194 | 5.49 | 1.7029 | 5.93 | 1.7800 |
| 4.62 | 1.5304 | 5.06 | 1.6214 | 5.50 | 1.7047 | 5.94 | 1.7817 |
| 4.63 | 1.5326 | 5.07 | 1.6233 | 5.51 | 1.7066 | 5.95 | 1.7834 |
| 4.64 | 1.5347 | 5.08 | 1.6253 | 5.52 | 1.7084 | 5.96 | 1.7851 |
| 4.65 | 1.5369 | 5.09 | 1.6273 | 5.53 | 1.7102 | 5.97 | 1.7867 |
| 4.66 | 1.5390 | 5.10 | 1.6292 | 5.54 | 1.7120 | 5.98 | 1.7884 |
| 4.67 | 1.5412 | 5.11 | 1.6312 | 5.55 | 1.7138 | 5.99 | 1.7901 |
| 4.68 | 1.5433 | 5.12 | 1.6332 | 5.56 | 1.7156 | 6.00 | 1.7918 |
| 4.69 | 1.5454 | 5.13 | 1.6351 | 5.57 | 1.7174 | 6.01 | 1.7934 |
| 4.70 | 1.5476 | 5.14 | 1.6371 | 5.58 | 1.7192 | 6.02 | 1.7951 |
| 4.71 | 1.5497 | 5.15 | 1.6390 | 5.59 | 1.7210 | 6.03 | 1.7967 |
| 4.72 | 1.5518 | 5.16 | 1.6409 | 5.60 | 1.7228 | 6.04 | 1.7984 |
| 4.73 | 1.5539 | 5.17 | 1.6429 | 5.61 | 1.7246 | 6.05 | 1.8001 |
| 4.74 | 1.5560 | 5.18 | 1.6448 | 5.62 | 1.7263 | 6.06 | 1.8017 |
| 4.75 | 1.5581 | 5.19 | 1.6467 | 5.63 | 1.7281 | 6.07 | 1.8034 |
| 4.76 | 1.5602 | 5.20 | 1.6487 | 5.64 | 1.7299 | 6.08 | 1.8050 |
| 4.77 | 1.5623 | 5.21 | 1.6506 | 5.65 | 1.7317 | 6.09 | 1.8066 |
| 4.78 | 1.5644 | 5.22 | 1.6525 | 5.66 | 1.7334 | 6.10 | 1.8083 |
| 4.79 | 1.5665 | 5.23 | 1.6544 | 5.67 | 1.7352 | 6.11 | 1.8099 |

| No. | Log | No. | Log | No. | Log | No. | Log |
|------|--------|------|--------|------|--------|------|--------|
| 6-12 | 1-8116 | 6-56 | 1-8810 | 7-00 | 1-9459 | 7-44 | 2-0069 |
| 6-13 | 1-8132 | 6-57 | 1-8825 | 7-01 | 1-9473 | 7-45 | 2-0082 |
| 6-14 | 1-8148 | 6-58 | 1-8840 | 7-02 | 1-9488 | 7-46 | 2-0096 |
| 6-15 | 1-8165 | 6-59 | 1-8856 | 7-03 | 1-9502 | 7-47 | 2-0109 |
| 6-16 | 1-8181 | 6-60 | 1-8871 | 7-04 | 1-9516 | 7-48 | 2-0122 |
| 6-17 | 1-8197 | 6-61 | 1-8886 | 7-05 | 1-9530 | 7-49 | 2-0136 |
| 6-18 | 1-8213 | 6-62 | 1-8901 | 7-06 | 1-9544 | 7-50 | 2-0149 |
| 6-19 | 1-8229 | 6-63 | 1-8916 | 7-07 | 1-9559 | 7-51 | 2-0162 |
| 6-20 | 1-8245 | 6-64 | 1-8931 | 7-08 | 1-9573 | 7-52 | 2-0176 |
| 6-21 | 1-8262 | 6-65 | 1-8946 | 7-09 | 1-9587 | 7-53 | 2-0189 |
| 6-22 | 1-8278 | 6-66 | 1-8961 | 7-10 | 1-9601 | 7-54 | 2-0202 |
| 6-23 | 1-8294 | 6-67 | 1-8976 | 7-11 | 1-9615 | 7-55 | 2-0215 |
| 6-24 | 1-8310 | 6-68 | 1-8991 | 7-12 | 1-9629 | 7-56 | 2-0229 |
| 6-25 | 1-8326 | 6-69 | 1-9006 | 7-13 | 1-9643 | 7-57 | 2-0242 |
| 6-26 | 1-8342 | 6-70 | 1-9021 | 7-14 | 1-9657 | 7-58 | 2-0255 |
| 6-27 | 1-8358 | 6-71 | 1-9036 | 7-15 | 1-9671 | 7-59 | 2-0268 |
| 6-28 | 1-8374 | 6-72 | 1-9051 | 7-16 | 1-9685 | 7-60 | 2-0281 |
| 6-29 | 1-8390 | 6-73 | 1-9066 | 7-17 | 1-9699 | 7-61 | 2-0295 |
| 6-30 | 1-8405 | 6-74 | 1-9081 | 7-18 | 1-9713 | 7-62 | 2-0308 |
| 6-31 | 1-8421 | 6-75 | 1-9095 | 7-19 | 1-9727 | 7-63 | 2-0321 |
| 6-32 | 1-8437 | 6-76 | 1-9110 | 7-20 | 1-9741 | 7-64 | 2-0334 |
| 6-33 | 1-8453 | 6-77 | 1-9125 | 7-21 | 1-9755 | 7-65 | 2-0347 |
| 6-34 | 1-8469 | 6-78 | 1-9140 | 7-22 | 1-9769 | 7-66 | 2-0360 |
| 6-35 | 1-8485 | 6-79 | 1-9155 | 7-23 | 1-9782 | 7-67 | 2-0373 |
| 6-36 | 1-8500 | 6-80 | 1-9169 | 7-24 | 1-9796 | 7-68 | 2-0386 |
| 6-37 | 1-8516 | 6-81 | 1-9184 | 7-25 | 1-9810 | 7-69 | 2-0399 |
| 6-38 | 1-8532 | 6-82 | 1-9199 | 7-26 | 1-9824 | 7-70 | 2-0412 |
| 6-39 | 1-8547 | 6-83 | 1-9213 | 7-27 | 1-9838 | 7-71 | 2-0425 |
| 6-40 | 1-8563 | 6-84 | 1-9228 | 7-28 | 1-9851 | 7-72 | 2-0438 |
| 6-41 | 1-8579 | 6-85 | 1-9242 | 7-29 | 1-9865 | 7-73 | 2-0451 |
| 6-42 | 1-8594 | 6-86 | 1-9257 | 7-30 | 1-9879 | 7-74 | 2-0464 |
| 6-43 | 1-8610 | 6-87 | 1-9272 | 7-31 | 1-9892 | 7-75 | 2-0477 |
| 6-44 | 1-8625 | 6-88 | 1-9286 | 7-32 | 1-9906 | 7-76 | 2-0490 |
| 6-45 | 1-8641 | 6-89 | 1-9301 | 7-33 | 1-9920 | 7-77 | 2-0503 |
| 6-46 | 1-8656 | 6-90 | 1-9315 | 7-34 | 1-9933 | 7-78 | 2-0516 |
| 6-47 | 1-8672 | 6-91 | 1-9330 | 7-35 | 1-9947 | 7-79 | 2-0528 |
| 6-48 | 1-8687 | 6-92 | 1-9344 | 7-36 | 1-9961 | 7-80 | 2-0541 |
| 6-49 | 1-8703 | 6-93 | 1-9359 | 7-37 | 1-9974 | 7-81 | 2-0554 |
| 6-50 | 1-8718 | 6-94 | 1-9373 | 7-38 | 1-9988 | 7-82 | 2-0567 |
| 6-51 | 1-8733 | 6-95 | 1-9387 | 7-39 | 2-0001 | 7-83 | 2-0580 |
| 6-52 | 1-8749 | 6-96 | 1-9402 | 7-40 | 2-0015 | 7-84 | 2-0592 |
| 6-53 | 1-8764 | 6-97 | 1-9416 | 7-41 | 2-0028 | 7-85 | 2-0605 |
| 6-54 | 1-8779 | 6-98 | 1-9430 | 7-42 | 2-0042 | 7-86 | 2-0618 |
| 6-55 | 1-8795 | 6-99 | 1-9445 | 7-43 | 2-0055 | 7-87 | 2-0631 |

| No. | Log | No. | Log. | No. | Log | No. | Log. |
|------|--------|------|--------|------|--------|------|--------|
| 7.88 | 2.0643 | 8.32 | 2.1187 | 8.76 | 2.1762 | 9.20 | 2.2192 |
| 7.89 | 2.0656 | 8.33 | 2.1199 | 8.77 | 2.1773 | 9.21 | 2.2203 |
| 7.90 | 2.0669 | 8.34 | 2.1211 | 8.78 | 2.1785 | 9.22 | 2.2214 |
| 7.91 | 2.0681 | 8.35 | 2.1223 | 8.79 | 2.1796 | 9.23 | 2.2225 |
| 7.92 | 2.0694 | 8.36 | 2.1235 | 8.80 | 2.1748 | 9.24 | 2.2235 |
| 7.93 | 2.0707 | 8.37 | 2.1247 | 8.81 | 2.1759 | 9.25 | 2.2246 |
| 7.94 | 2.0719 | 8.38 | 2.1258 | 8.82 | 2.1770 | 9.26 | 2.2257 |
| 7.95 | 2.0732 | 8.39 | 2.1270 | 8.83 | 2.1782 | 9.27 | 2.2268 |
| 7.96 | 2.0744 | 8.40 | 2.1282 | 8.84 | 2.1793 | 9.28 | 2.2279 |
| 7.97 | 2.0757 | 8.41 | 2.1294 | 8.85 | 2.1804 | 9.29 | 2.2289 |
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| 7.99 | 2.0782 | 8.43 | 2.1318 | 8.87 | 2.1827 | 9.31 | 2.2311 |
| 8.00 | 2.0794 | 8.44 | 2.1330 | 8.88 | 2.1838 | 9.32 | 2.2322 |
| 8.01 | 2.0807 | 8.45 | 2.1342 | 8.89 | 2.1849 | 9.33 | 2.2332 |
| 8.02 | 2.0819 | 8.46 | 2.1353 | 8.90 | 2.1861 | 9.34 | 2.2343 |
| 8.03 | 2.0832 | 8.47 | 2.1365 | 8.91 | 2.1872 | 9.35 | 2.2354 |
| 8.04 | 2.0844 | 8.48 | 2.1377 | 8.92 | 2.1883 | 9.36 | 2.2364 |
| 8.05 | 2.0857 | 8.49 | 2.1389 | 8.93 | 2.1894 | 9.37 | 2.2375 |
| 8.06 | 2.0869 | 8.50 | 2.1401 | 8.94 | 2.1905 | 9.38 | 2.2386 |
| 8.07 | 2.0882 | 8.51 | 2.1412 | 8.95 | 2.1917 | 9.39 | 2.2396 |
| 8.08 | 2.0894 | 8.52 | 2.1424 | 8.96 | 2.1928 | 9.40 | 2.2407 |
| 8.09 | 2.0906 | 8.53 | 2.1436 | 8.97 | 2.1939 | 9.41 | 2.2418 |
| 8.10 | 2.0919 | 8.54 | 2.1448 | 8.98 | 2.1950 | 9.42 | 2.2428 |
| 8.11 | 2.0931 | 8.55 | 2.1459 | 8.99 | 2.1961 | 9.43 | 2.2439 |
| 8.12 | 2.0943 | 8.56 | 2.1471 | 9.00 | 2.1972 | 9.44 | 2.2450 |
| 8.13 | 2.0956 | 8.57 | 2.1483 | 9.01 | 2.1983 | 9.45 | 2.2460 |
| 8.14 | 2.0968 | 8.58 | 2.1494 | 9.02 | 2.1994 | 9.46 | 2.2471 |
| 8.15 | 2.0980 | 8.59 | 2.1506 | 9.03 | 2.2006 | 9.47 | 2.2481 |
| 8.16 | 2.0992 | 8.60 | 2.1518 | 9.04 | 2.2017 | 9.48 | 2.2492 |
| 8.17 | 2.1005 | 8.61 | 2.1529 | 9.05 | 2.2028 | 9.49 | 2.2502 |
| 8.18 | 2.1017 | 8.62 | 2.1541 | 9.06 | 2.2039 | 9.50 | 2.2513 |
| 8.19 | 2.1029 | 8.63 | 2.1552 | 9.07 | 2.2050 | 9.51 | 2.2523 |
| 8.20 | 2.1041 | 8.64 | 2.1564 | 9.08 | 2.2061 | 9.52 | 2.2534 |
| 8.21 | 2.1054 | 8.65 | 2.1576 | 9.09 | 2.2072 | 9.53 | 2.2544 |
| 8.22 | 2.1066 | 8.66 | 2.1587 | 9.10 | 2.2083 | 9.54 | 2.2555 |
| 8.23 | 2.1078 | 8.67 | 2.1599 | 9.11 | 2.2094 | 9.55 | 2.2565 |
| 8.24 | 2.1090 | 8.68 | 2.1610 | 9.12 | 2.2105 | 9.56 | 2.2576 |
| 8.25 | 2.1102 | 8.69 | 2.1622 | 9.13 | 2.2116 | 9.57 | 2.2586 |
| 8.26 | 2.1114 | 8.70 | 2.1633 | 9.14 | 2.2127 | 9.58 | 2.2597 |
| 8.27 | 2.1126 | 8.71 | 2.1645 | 9.15 | 2.2138 | 9.59 | 2.2607 |
| 8.28 | 2.1138 | 8.72 | 2.1656 | 9.16 | 2.2148 | 9.60 | 2.2618 |
| 8.29 | 2.1150 | 8.73 | 2.1668 | 9.17 | 2.2159 | 9.61 | 2.2628 |
| 8.30 | 2.1163 | 8.74 | 2.1679 | 9.18 | 2.2170 | 9.62 | 2.2638 |
| 8.31 | 2.1175 | 8.75 | 2.1691 | 9.19 | 2.2181 | 9.63 | 2.2649 |

| No. | Log. | No. | Log. | No. | Log. | No. | Log. |
|------|--------|-------|--------|-------|--------|-------|--------|
| 9-64 | 2-2659 | 9-89 | 2-2915 | 13-50 | 2-6027 | 29-00 | 3-3673 |
| 9-65 | 2-2670 | 9-90 | 2-2925 | 13-75 | 2-6211 | 30-00 | 3-4012 |
| 9-66 | 2-2680 | 9-91 | 2-2935 | 14-00 | 2-6391 | 31-00 | 3-4340 |
| 9-67 | 2-2690 | 9-92 | 2-2946 | 14-25 | 2-6567 | 32-00 | 3-4657 |
| 9-68 | 2-2701 | 9-93 | 2-2956 | 14-50 | 2-6740 | 33-00 | 3-4965 |
| 9-69 | 2-2711 | 9-94 | 2-2966 | 14-75 | 2-6913 | 34-00 | 3-5263 |
| 9-70 | 2-2721 | 9-95 | 2-2976 | 15-00 | 2-7084 | 35-00 | 3-5553 |
| 9-71 | 2-2732 | 9-96 | 2-2986 | 15-50 | 2-7408 | 36-00 | 3-5835 |
| 9-72 | 2-2742 | 9-97 | 2-2996 | 16-00 | 2-7726 | 37-00 | 3-6109 |
| 9-73 | 2-2752 | 9-98 | 2-3006 | 16-50 | 2-8034 | 38-00 | 3-6376 |
| 9-74 | 2-2762 | 9-99 | 2-3016 | 17-00 | 2-8332 | 39-00 | 3-6636 |
| 9-75 | 2-2773 | 10-00 | 2-3026 | 17-50 | 2-8621 | 40-00 | 3-6889 |
| 9-76 | 2-2783 | 10-25 | 2-3279 | 18-00 | 2-8904 | 41-00 | 3-7136 |
| 9-77 | 2-2793 | 10-50 | 2-3513 | 18-50 | 2-9173 | 42-00 | 3-7377 |
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| 9-79 | 2-2814 | 11-00 | 2-3979 | 19-50 | 2-9703 | 44-00 | 3-7842 |
| 9-80 | 2-2824 | 11-25 | 2-4201 | 20-00 | 2-9957 | 45-00 | 3-8067 |
| 9-81 | 2-2834 | 11-50 | 2-4430 | 21-00 | 3-0115 | 46-00 | 3-8286 |
| 9-82 | 2-2844 | 11-75 | 2-4636 | 22-00 | 3-0911 | 47-00 | 3-8501 |
| 9-83 | 2-2854 | 12-00 | 2-4849 | 23-00 | 3-1355 | 48-00 | 3-8712 |
| 9-84 | 2-2865 | 12-25 | 2-5052 | 24-00 | 3-1781 | 49-00 | 3-8918 |
| 9-85 | 2-2875 | 12-50 | 2-5262 | 25-00 | 3-2189 | 50-00 | 3-9120 |
| 9-86 | 2-2885 | 12-75 | 2-5455 | 26-00 | 3-2581 | | |
| 9-87 | 2-2895 | 13-00 | 2-5649 | 27-00 | 3-2958 | | |
| 9-88 | 2-2905 | 13-25 | 2-5840 | 28-00 | 3-3322 | | |

TABLES
LOGARITHMS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----|------|------|------|------|------|------|------|------|------|------|---|---|----|----|----|----|----|----|----|
| 10 | 0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4 | 9 | 13 | 17 | 21 | 26 | 30 | 34 | 38 |
| 11 | 0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 37 |
| 12 | 0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1039 | 1072 | 1106 | 4 | 7 | 11 | 15 | 19 | 23 | 27 | 31 | 35 |
| 13 | 1134 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3 | 7 | 10 | 14 | 18 | 22 | 26 | 30 | 33 |
| 14 | 1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 28 |
| 15 | 1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3 | 6 | 9 | 11 | 14 | 17 | 20 | 23 | 26 |
| 16 | 2031 | 2065 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3 | 5 | 8 | 11 | 14 | 16 | 19 | 22 | 24 |
| 17 | 2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 |
| 18 | 2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2 | 5 | 7 | 10 | 12 | 14 | 16 | 19 | 21 |
| 19 | 2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| 20 | 3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| 21 | 3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 22 | 3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3598 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 23 | 3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 24 | 3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |

| | 397 | 401 | 408 | 404 | 406 | 408 | 409 | 411 | 413 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
|----|------|------|------|------|------|------|------|------|------|----|----|---|---|---|----|----|----|----|
| 25 | 397 | 401 | 408 | 404 | 406 | 408 | 409 | 411 | 413 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
| 26 | 4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 23 | 23 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| 27 | 4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4441 | 23 | 23 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 28 | 4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 23 | 23 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 29 | 4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 13 | 13 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 30 | 4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4885 | 13 | 13 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 31 | 4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 13 | 13 | 4 | 6 | 7 | 8 | 10 | 11 | 12 |
| 32 | 5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 13 | 13 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 33 | 5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 13 | 13 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 34 | 5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 13 | 13 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 35 | 5441 | 5453 | 5465 | 5477 | 5490 | 5502 | 5514 | 5527 | 5539 | 12 | 12 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 36 | 5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 12 | 12 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 37 | 5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 12 | 12 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 38 | 5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 12 | 12 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 39 | 5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 12 | 12 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 40 | 6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 12 | 12 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| 41 | 6129 | 6139 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 12 | 12 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 42 | 6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 12 | 12 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 43 | 6337 | 6347 | 6357 | 6367 | 6375 | 6385 | 6395 | 6405 | 6415 | 12 | 12 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 44 | 6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 12 | 12 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 45 | 6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 12 | 12 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 46 | 6625 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 12 | 12 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 47 | 6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 12 | 12 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 12 | 12 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 12 | 12 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 50 | 6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 12 | 12 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |

LOGARITHMS—continued

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----|------|------|------|------|------|------|------|------|------|---|---|---|---|---|---|---|---|---|---|
| 51 | 7076 | 7084 | 7093 | 7101 | 7110 | 7126 | 7135 | 7143 | 7152 | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
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| 53 | 7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7316 | | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
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| 55 | 7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7474 | | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 56 | 7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7551 | | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 57 | 7559 | 7566 | 7574 | 7582 | 7590 | 7597 | 7604 | 7612 | 7627 | | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 58 | 7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7701 | | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 59 | 7709 | 7716 | 7723 | 7731 | 7739 | 7745 | 7752 | 7760 | 7774 | | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 60 | 7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7846 | | 1 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 6 |
| 61 | 7853 | 7860 | 7866 | 7875 | 7882 | 7889 | 7896 | 7903 | 7917 | | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 62 | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7987 | | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8055 | | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8090 | 8096 | 8102 | 8109 | 8122 | | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 65 | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8189 | | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 66 | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8254 | | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 67 | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8319 | | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 68 | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8382 | | 1 | 1 | 2 | 3 | 3 | 4 | 1 | 5 | 6 |
| 69 | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8445 | | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 70 | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8506 | | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 71 | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8567 | | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 72 | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8627 | | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 73 | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8686 | | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 74 | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8745 | | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |

| | | | | | | | | | | | | | | | | | | | | | |
|----|------|------|------|------|------|------|------|------|------|------|---|---|---|---|---|---|---|---|---|---|---|
| 75 | 8751 | 8756 | 8762 | 8766 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 76 | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 77 | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 78 | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 79 | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 80 | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 2 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 81 | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 82 | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 83 | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 84 | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 85 | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 86 | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 87 | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 88 | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 89 | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 90 | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 91 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9629 | 9634 | 9639 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 92 | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 93 | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 94 | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 95 | 9777 | 9782 | 9786 | 9791 | 9796 | 9800 | 9805 | 9809 | 9814 | 9819 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 96 | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 97 | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 98 | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 99 | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |

ANALOGATHEMS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|------|------|------|------|------|------|------|------|------|------|---|---|---|---|---|---|---|---|---|
| .00 | 1000 | 1002 | 1005 | 1007 | 1009 | 1012 | 1014 | 1016 | 1019 | 1021 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .01 | 1023 | 1026 | 1028 | 1030 | 1033 | 1035 | 1038 | 1040 | 1042 | 1045 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .02 | 1047 | 1050 | 1052 | 1054 | 1057 | 1059 | 1062 | 1064 | 1067 | 1067 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .03 | 1072 | 1074 | 1076 | 1079 | 1081 | 1084 | 1086 | 1089 | 1091 | 1094 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .04 | 1096 | 1099 | 1102 | 1104 | 1107 | 1109 | 1112 | 1114 | 1117 | 1119 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .05 | 1122 | 1125 | 1127 | 1130 | 1132 | 1135 | 1138 | 1140 | 1143 | 1146 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .06 | 1148 | 1151 | 1153 | 1156 | 1159 | 1161 | 1164 | 1167 | 1169 | 1172 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .07 | 1175 | 1178 | 1180 | 1183 | 1186 | 1189 | 1191 | 1194 | 1197 | 1199 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .08 | 1202 | 1205 | 1208 | 1211 | 1213 | 1216 | 1219 | 1222 | 1225 | 1227 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .09 | 1230 | 1233 | 1236 | 1239 | 1242 | 1245 | 1247 | 1250 | 1253 | 1256 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .10 | 1259 | 1262 | 1265 | 1268 | 1271 | 1274 | 1276 | 1279 | 1282 | 1285 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .11 | 1288 | 1291 | 1294 | 1297 | 1300 | 1303 | 1306 | 1309 | 1312 | 1315 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .12 | 1318 | 1321 | 1324 | 1327 | 1330 | 1334 | 1337 | 1340 | 1343 | 1346 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .13 | 1349 | 1352 | 1355 | 1358 | 1361 | 1365 | 1368 | 1371 | 1374 | 1377 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .14 | 1380 | 1384 | 1387 | 1390 | 1393 | 1396 | 1400 | 1403 | 1406 | 1409 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .15 | 1413 | 1416 | 1419 | 1422 | 1426 | 1429 | 1432 | 1435 | 1439 | 1442 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .16 | 1445 | 1449 | 1452 | 1455 | 1459 | 1462 | 1466 | 1469 | 1472 | 1476 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .17 | 1479 | 1483 | 1486 | 1489 | 1493 | 1496 | 1500 | 1503 | 1507 | 1510 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .18 | 1514 | 1517 | 1521 | 1524 | 1528 | 1531 | 1535 | 1538 | 1542 | 1545 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .19 | 1549 | 1552 | 1556 | 1560 | 1563 | 1567 | 1570 | 1574 | 1578 | 1581 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .20 | 1585 | 1589 | 1592 | 1596 | 1600 | 1603 | 1607 | 1611 | 1614 | 1618 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .21 | 1622 | 1626 | 1629 | 1633 | 1637 | 1641 | 1644 | 1648 | 1652 | 1656 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .22 | 1660 | 1663 | 1667 | 1671 | 1675 | 1679 | 1683 | 1687 | 1690 | 1694 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .23 | 1698 | 1702 | 1706 | 1710 | 1714 | 1718 | 1722 | 1726 | 1730 | 1734 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| .24 | 1738 | 1742 | 1746 | 1750 | 1754 | 1758 | 1762 | 1766 | 1770 | 1774 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |

| | | | | | | | | | | | | | | | | | | | |
|----|------|------|------|------|------|------|------|------|------|------|---|---|---|---|---|---|---|---|---|
| 25 | 1778 | 1782 | 1786 | 1791 | 1795 | 1799 | 1803 | 1807 | 1811 | 1816 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| 26 | 1520 | 1824 | 1826 | 1832 | 1837 | 1841 | 1845 | 1849 | 1854 | 1858 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 27 | 1862 | 1866 | 1871 | 1875 | 1879 | 1884 | 1888 | 1892 | 1897 | 1901 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 28 | 1905 | 1910 | 1914 | 1919 | 1923 | 1928 | 1932 | 1936 | 1941 | 1945 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 29 | 1950 | 1954 | 1959 | 1963 | 1968 | 1972 | 1977 | 1982 | 1986 | 1991 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 30 | 1995 | 2000 | 2004 | 2009 | 2014 | 2018 | 2023 | 2028 | 2032 | 2037 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 31 | 2042 | 2046 | 2051 | 2056 | 2061 | 2065 | 2070 | 2075 | 2080 | 2084 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 32 | 2089 | 2094 | 2099 | 2104 | 2109 | 2113 | 2118 | 2123 | 2128 | 2133 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 33 | 2138 | 2143 | 2148 | 2153 | 2158 | 2163 | 2168 | 2173 | 2178 | 2183 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| 34 | 2188 | 2193 | 2198 | 2203 | 2208 | 2213 | 2218 | 2223 | 2228 | 2234 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 35 | 2239 | 2244 | 2249 | 2254 | 2259 | 2265 | 2270 | 2275 | 2280 | 2286 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 36 | 2291 | 2296 | 2301 | 2307 | 2312 | 2317 | 2323 | 2328 | 2333 | 2339 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 37 | 2344 | 2350 | 2355 | 2360 | 2366 | 2371 | 2377 | 2382 | 2388 | 2393 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 38 | 2399 | 2404 | 2410 | 2415 | 2421 | 2427 | 2432 | 2438 | 2443 | 2450 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 39 | 2455 | 2460 | 2466 | 2472 | 2477 | 2483 | 2489 | 2495 | 2500 | 2506 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 40 | 2512 | 2518 | 2523 | 2529 | 2535 | 2541 | 2547 | 2553 | 2559 | 2564 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 41 | 2570 | 2576 | 2582 | 2588 | 2594 | 2600 | 2606 | 2612 | 2618 | 2624 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 42 | 2630 | 2636 | 2642 | 2649 | 2655 | 2661 | 2667 | 2673 | 2679 | 2685 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 43 | 2692 | 2698 | 2704 | 2710 | 2716 | 2723 | 2729 | 2735 | 2742 | 2748 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 44 | 2754 | 2761 | 2767 | 2773 | 2780 | 2786 | 2793 | 2799 | 2805 | 2812 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 45 | 2818 | 2825 | 2831 | 2838 | 2844 | 2851 | 2858 | 2864 | 2871 | 2877 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 46 | 2884 | 2891 | 2897 | 2904 | 2911 | 2917 | 2924 | 2931 | 2938 | 2944 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 47 | 2951 | 2958 | 2965 | 2972 | 2979 | 2985 | 2992 | 2999 | 3006 | 3013 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 48 | 3020 | 3027 | 3034 | 3041 | 3048 | 3055 | 3062 | 3069 | 3076 | 3083 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |
| 49 | 3090 | 3097 | 3105 | 3112 | 3119 | 3126 | 3133 | 3141 | 3148 | 3155 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |

ANTILOGARITHMS—continued

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|------|------|------|------|------|------|------|------|------|------|---|---|---|---|---|---|---|---|---|
| .50 | 3163 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| .51 | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| .52 | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| .53 | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| .54 | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| .55 | 3548 | 3556 | 3565 | 3573 | 3581 | 3580 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| .56 | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 3698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 |
| .57 | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 |
| .58 | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 5 | 6 | 7 |
| .59 | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |
| .60 | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |
| .61 | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |
| .62 | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |
| .63 | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |
| .64 | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |
| .65 | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |
| .66 | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |
| .67 | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 5 | 5 | 6 | 7 |
| .68 | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 6 | 6 | 7 | 8 |
| .69 | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 6 | 6 | 7 | 8 |
| .70 | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 6 | 6 | 7 | 8 |
| .71 | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 6 | 6 | 7 | 8 |
| .72 | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 6 | 6 | 7 | 8 |
| .73 | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 3 | 4 | 5 | 6 | 6 | 6 | 7 | 8 |
| .74 | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 6 | 6 | 7 | 8 |

| | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 |
|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|----|
| 56.0 | 5754 | 5888 | 6026 | 6166 | 6310 | 6457 | 6607 | 6761 | 6918 | 7079 | 7244 | 7413 | 7586 | 7762 | 7943 | 8128 | 8318 | 8511 | 8710 | 8913 | 9120 | 9333 | 9550 | 9772 | |
| 56.1 | 5761 | 5895 | 6033 | 6173 | 6317 | 6464 | 6614 | 6767 | 6924 | 7084 | 7248 | 7417 | 7590 | 7767 | 7948 | 8133 | 8323 | 8516 | 8715 | 8918 | 9125 | 9338 | 9555 | 9778 | |
| 56.2 | 5768 | 5902 | 6040 | 6180 | 6324 | 6471 | 6621 | 6773 | 6930 | 7090 | 7254 | 7423 | 7596 | 7773 | 7954 | 8139 | 8329 | 8522 | 8721 | 8924 | 9131 | 9344 | 9562 | 9785 | |
| 56.3 | 5775 | 5909 | 6047 | 6187 | 6331 | 6478 | 6628 | 6780 | 6937 | 7097 | 7261 | 7430 | 7603 | 7780 | 7961 | 8146 | 8336 | 8529 | 8728 | 8931 | 9138 | 9351 | 9570 | 9793 | |
| 56.4 | 5782 | 5916 | 6054 | 6194 | 6338 | 6485 | 6635 | 6787 | 6944 | 7104 | 7268 | 7437 | 7610 | 7787 | 7968 | 8153 | 8343 | 8536 | 8735 | 8938 | 9145 | 9358 | 9577 | 9800 | |
| 56.5 | 5789 | 5923 | 6061 | 6201 | 6345 | 6492 | 6642 | 6794 | 6951 | 7111 | 7275 | 7444 | 7617 | 7794 | 7975 | 8160 | 8350 | 8543 | 8742 | 8945 | 9152 | 9365 | 9584 | 9807 | |
| 56.6 | 5796 | 5930 | 6068 | 6208 | 6352 | 6499 | 6649 | 6801 | 6958 | 7118 | 7282 | 7451 | 7624 | 7801 | 7982 | 8167 | 8357 | 8550 | 8749 | 8952 | 9159 | 9372 | 9591 | 9814 | |
| 56.7 | 5803 | 5937 | 6075 | 6215 | 6359 | 6506 | 6656 | 6808 | 6965 | 7125 | 7289 | 7458 | 7631 | 7808 | 7989 | 8174 | 8364 | 8557 | 8756 | 8959 | 9166 | 9379 | 9598 | 9821 | |
| 56.8 | 5810 | 5944 | 6082 | 6222 | 6366 | 6513 | 6663 | 6815 | 6972 | 7132 | 7296 | 7465 | 7638 | 7815 | 7996 | 8181 | 8371 | 8564 | 8763 | 8966 | 9173 | 9386 | 9605 | 9828 | |
| 56.9 | 5817 | 5951 | 6089 | 6229 | 6373 | 6520 | 6670 | 6822 | 6979 | 7139 | 7303 | 7472 | 7645 | 7822 | 8003 | 8188 | 8378 | 8571 | 8770 | 8973 | 9180 | 9393 | 9612 | 9835 | |
| 57.0 | 5824 | 5958 | 6096 | 6236 | 6380 | 6527 | 6677 | 6829 | 6986 | 7146 | 7310 | 7479 | 7652 | 7829 | 8010 | 8195 | 8385 | 8578 | 8777 | 8980 | 9187 | 9400 | 9619 | 9842 | |
| 57.1 | 5831 | 5965 | 6103 | 6243 | 6387 | 6534 | 6684 | 6836 | 6993 | 7153 | 7317 | 7486 | 7659 | 7836 | 8017 | 8202 | 8392 | 8585 | 8784 | 8987 | 9194 | 9407 | 9626 | 9849 | |
| 57.2 | 5838 | 5972 | 6110 | 6250 | 6394 | 6541 | 6691 | 6843 | 6999 | 7159 | 7323 | 7492 | 7665 | 7842 | 8023 | 8208 | 8398 | 8591 | 8790 | 8993 | 9200 | 9413 | 9632 | 9855 | |
| 57.3 | 5845 | 5979 | 6117 | 6257 | 6401 | 6548 | 6698 | 6850 | 7006 | 7166 | 7330 | 7500 | 7673 | 7850 | 8031 | 8216 | 8406 | 8599 | 8798 | 8999 | 9206 | 9419 | 9638 | 9861 | |
| 57.4 | 5852 | 5986 | 6124 | 6264 | 6408 | 6555 | 6705 | 6857 | 7013 | 7173 | 7337 | 7507 | 7680 | 7857 | 8038 | 8223 | 8413 | 8606 | 8805 | 9006 | 9213 | 9426 | 9645 | 9868 | |
| 57.5 | 5859 | 5993 | 6131 | 6271 | 6415 | 6562 | 6712 | 6864 | 7020 | 7180 | 7344 | 7514 | 7687 | 7864 | 8045 | 8230 | 8420 | 8613 | 8812 | 9013 | 9220 | 9433 | 9652 | 9875 | |
| 57.6 | 5866 | 6000 | 6138 | 6278 | 6422 | 6569 | 6719 | 6871 | 7027 | 7187 | 7351 | 7521 | 7694 | 7871 | 8052 | 8237 | 8427 | 8620 | 8819 | 9020 | 9227 | 9440 | 9659 | 9882 | |
| 57.7 | 5873 | 6007 | 6145 | 6285 | 6429 | 6576 | 6726 | 6878 | 7034 | 7194 | 7358 | 7528 | 7699 | 7876 | 8057 | 8242 | 8432 | 8625 | 8824 | 9025 | 9232 | 9445 | 9664 | 9887 | |
| 57.8 | 5880 | 6014 | 6152 | 6292 | 6436 | 6583 | 6733 | 6885 | 7041 | 7201 | 7365 | 7535 | 7706 | 7883 | 8064 | 8249 | 8439 | 8632 | 8831 | 9032 | 9239 | 9452 | 9671 | 9894 | |
| 57.9 | 5887 | 6021 | 6159 | 6299 | 6443 | 6590 | 6740 | 6892 | 7048 | 7208 | 7372 | 7542 | 7713 | 7890 | 8071 | 8256 | 8446 | 8639 | 8838 | 9039 | 9246 | 9459 | 9678 | 9901 | |
| 58.0 | 5894 | 6028 | 6166 | 6306 | 6450 | 6597 | 6747 | 6899 | 7055 | 7215 | 7379 | 7549 | 7720 | 7897 | 8078 | 8263 | 8453 | 8646 | 8845 | 9046 | 9253 | 9466 | 9685 | 9908 | |
| 58.1 | 5901 | 6035 | 6173 | 6313 | 6457 | 6604 | 6754 | 6906 | 7062 | 7222 | 7386 | 7556 | 7727 | 7904 | 8085 | 8270 | 8460 | 8653 | 8852 | 9053 | 9260 | 9473 | 9692 | 9915 | |
| 58.2 | 5908 | 6042 | 6180 | 6320 | 6464 | 6611 | 6761 | 6913 | 7069 | 7229 | 7393 | 7563 | 7734 | 7911 | 8092 | 8277 | 8467 | 8660 | 8859 | 9060 | 9267 | 9480 | 9700 | 9923 | |
| 58.3 | 5915 | 6049 | 6187 | 6327 | 6471 | 6618 | 6768 | 6920 | 7076 | 7236 | 7400 | 7569 | 7740 | 7917 | 8098 | 8283 | 8473 | 8666 | 8865 | 9066 | 9273 | 9486 | 9706 | 9929 | |
| 58.4 | 5922 | 6056 | 6194 | 6334 | 6478 | 6625 | 6775 | 6927 | 7083 | 7243 | 7407 | 7576 | 7747 | 7924 | 8105 | 8290 | 8480 | 8673 | 8872 | 9073 | 9280 | 9493 | 9713 | 9936 | |
| 58.5 | 5929 | 6063 | 6201 | 6341 | 6485 | 6632 | 6782 | 6934 | 7090 | 7250 | 7414 | 7583 | 7754 | 7931 | 8112 | 8297 | 8487 | 8680 | 8879 | 9079 | 9286 | 9499 | 9719 | 9942 | |
| 58.6 | 5936 | 6070 | 6208 | 6348 | 6492 | 6639 | 6789 | 6941 | 7097 | 7257 | 7421 | 7590 | 7761 | 7938 | 8119 | 8304 | 8494 | 8687 | 8886 | 9086 | 9293 | 9506 | 9726 | 9949 | |
| 58.7 | 5943 | 6077 | 6215 | 6355 | 6499 | 6646 | 6796 | 6948 | 7104 | 7264 | 7428 | 7597 | 7768 | 7945 | 8126 | 8311 | 8499 | 8692 | 8891 | 9091 | 9298 | 9511 | 9731 | 9954 | |
| 58.8 | 5950 | 6084 | 6222 | 6362 | 6506 | 6653 | 6803 | 6955 | 7111 | 7271 | 7435 | 7604 | 7775 | 7952 | 8133 | 8318 | 8506 | 8699 | 8898 | 9098 | 9305 | 9518 | 9738 | 9961 | |
| 58.9 | 5957 | 6091 | 6229 | 6369 | 6513 | 6660 | 6810 | 6962 | 7118 | 7278 | 7442 | 7611 | 7782 | 7959 | 8140 | 8325 | 8513 | 8706 | 8905 | 9105 | 9312 | 9525 | 9745 | 9968 | |
| 59.0 | 5964 | 6098 | 6236 | 6376 | 6520 | 6667 | 6817 | 6969 | 7125 | 7285 | 7449 | 7618 | 7789 | 7966 | 8147 | 8332 | 8520 | 8713 | 8912 | 9112 | 9319 | 9532 | 9752 | 9975 | |
| 59.1 | 5971 | 6105 | 6243 | 6383 | 6527 | 6674 | 6824 | 6976 | 7132 | 7292 | 7456 | 7625 | 7796 | 7973 | 8154 | 8339 | 8527 | 8720 | 8919 | 9119 | 9326 | 9539 | 9759 | 9982 | |
| 59.2 | 5978 | 6112 | 6250 | 6390 | 6534 | 6681 | 6831 | 6983 | 7139 | 7299 | 7463 | 7632 | 7803 | 7980 | 8161 | 8346 | 8534 | 8727 | 8926 | 9126 | 9333 | 9546 | 9766 | 9989 | |
| 59.3 | 5985 | 6119 | 6257 | 6397 | 6541 | 6688 | 6838 | 6990 | 7146 | 7306 | 7470 | 7639 | 7810 | 7987 | 8168 | 8353 | 8541 | 8734 | 8933 | 9133 | 9340 | 9553 | 9773 | 9996 | |
| 59.4 | 5992 | 6126 | 6264 | 6404 | 6548 | 6695 | 6845 | 6997 | 7153 | 7313 | 7477 | 7646 | 7817 | 7994 | 8175 | 8360 | 8548 | 8741 | 8940 | 9139 | 9346 | 9559 | 9779 | 9999 | |
| 59.5 | 5999 | 6133 | 6271 | 6411 | 6555 | 6702 | 6852 | 6999 | 7155 | 7315 | 7479 | 7648 | 7819 | 7996 | 8177 | 8362 | 8550 | 8743 | 8942 | 9141 | 9348 | 9561 | 9781 | 9999 | |
| 59.6 | 6006 | 6140 | 6278 | 6418 | 6562 | 6709 | 6859 | 7006 | 7162 | 7322 | 7486 | 7655 | 7826 | 7999 | 8180 | 8365 | 8553 | 8746 | 8945 | 9144 | 9351 | 9564 | 9784 | 9999 | |
| 59.7 | 6013 | 6147 | 6285 | 6425 | 6569 | 6716 | 6866 | 7013 | 7169 | 7329 | 7493 | 7662 | 7833 | 7999 | 8180 | 8365 | 8553 | 8746 | 8945 | 9144 | 9351 | 9564 | 9784 | 9999 | |
| 59.8 | 6020 | 6154 | 6292 | 6432 | 6576 | 6723 | 6873 | 7020 | 7176 | 7336 | 7500 | 7669 | 7840 | 8006 | 8187 | 8372 | 8560 | 8753 | 8952 | 9151 | 9358 | 9571 | 9791 | 9999 | |
| 59.9 | 6027 | 6161 | 6299 | 6439 | 6583 | 6730 | 6880 | 7027 | 7183 | 7343 | 7507 | 7676 | 7847 | 8013 | 8194 | 8379 | 8567 | 8760 | 8959 | 9158 | 9365 | 9578 | 9798 | 9999 | |

TRIGONOMETRICAL FUNCTIONS.

| Angle. | | Chord. | Sine. | Tangent. | Co tangent. | Co-ine. | | |
|----------|----------|--------|-------|----------|-------------|---------|-------|--------|
| Degrees. | Radians. | | | | | | | |
| 0° | 0 | 0 | 0 | 0 | ∞ | 1 | 1.414 | 1.5708 |
| 1 | 0.0175 | 0.17 | 0.175 | 0.175 | 57.2900 | .9998 | 1.402 | 1.5588 |
| 2 | 0.0349 | 0.35 | 0.349 | 0.349 | 28.6363 | .9994 | 1.389 | 1.5359 |
| 3 | 0.0524 | 0.52 | 0.523 | 0.524 | 19.0811 | .9986 | 1.377 | 1.5184 |
| 4 | 0.0698 | 0.70 | 0.698 | 0.699 | 14.3007 | .9976 | 1.364 | 1.5010 |
| 5 | 0.0873 | 0.87 | 0.872 | 0.875 | 11.4301 | .9962 | 1.351 | 1.4835 |
| 6 | 0.1047 | 1.05 | 1.045 | 1.051 | 9.5144 | .9945 | 1.336 | 1.4661 |
| 7 | 0.1222 | 1.22 | 1.219 | 1.228 | 8.1443 | .9925 | 1.325 | 1.4486 |
| 8 | 0.1396 | 1.40 | 1.392 | 1.405 | 7.1154 | .9903 | 1.312 | 1.4312 |
| 9 | 0.1571 | 1.57 | 1.564 | 1.584 | 6.3188 | .9877 | 1.299 | 1.4137 |
| 10 | 0.1745 | 1.74 | 1.736 | 1.763 | 5.6713 | .9848 | 1.286 | 1.3963 |
| 11 | 0.1920 | 1.92 | 1.908 | 1.914 | 5.1446 | .9816 | 1.272 | 1.3788 |
| 12 | 0.2094 | 2.09 | 2.079 | 2.126 | 4.7046 | .9781 | 1.259 | 1.3614 |
| 13 | 0.2269 | 2.26 | 2.250 | 2.309 | 4.3315 | .9744 | 1.245 | 1.3439 |
| 14 | 0.2443 | 2.44 | 2.419 | 2.493 | 4.0108 | .9703 | 1.231 | 1.3265 |
| 15 | 0.2618 | 2.61 | 2.588 | 2.679 | 3.7321 | .9659 | 1.216 | 1.3090 |
| 16 | 0.2793 | 2.78 | 2.756 | 2.867 | 3.4874 | .9613 | 1.204 | 1.2915 |
| 17 | 0.2967 | 2.96 | 2.924 | 3.057 | 3.2709 | .9563 | 1.190 | 1.2741 |
| 18 | 0.3142 | 3.13 | 3.090 | 3.249 | 3.0777 | .9511 | 1.176 | 1.2566 |
| 19 | 0.3316 | 3.30 | 3.256 | 3.443 | 2.9042 | .9455 | 1.161 | 1.2392 |
| 20 | 0.3491 | 3.47 | 3.420 | 3.640 | 2.7475 | .9397 | 1.147 | 1.2217 |
| 21 | 0.3665 | 3.64 | 3.584 | 3.839 | 2.6051 | .9336 | 1.133 | 1.2043 |
| 22 | 0.3840 | 3.82 | 3.746 | 4.040 | 2.4751 | .9272 | 1.118 | 1.1868 |
| 23 | 0.4014 | 3.99 | 3.907 | 4.245 | 2.3559 | .9205 | 1.104 | 1.1694 |
| 24 | 0.4189 | 4.16 | 4.067 | 4.452 | 2.2460 | .9135 | 1.089 | 1.1519 |

[illegible]

TABLE OF SQUARES, CUBES, ETC.

| No. | Square. | Cube. | Square Root. | Cube Root. | No. | Square. | Cube. | Square Root. | Cube Root. |
|-----|---------|---------|--------------|------------|------|---------|----------|--------------|------------|
| 1 | 1 | 1 | 1 | 1 | 6 | 36 | 216 | 6 | 1.8171 |
| 1.1 | 1.21 | 1.331 | 1.0488 | 1.0323 | 6.1 | 37.21 | 226.981 | 2.4698 | 1.8272 |
| 1.2 | 1.44 | 1.728 | 1.0955 | 1.0627 | 6.2 | 38.44 | 238.328 | 2.49 | 1.8371 |
| 1.3 | 1.69 | 2.197 | 1.1402 | 1.0913 | 6.3 | 39.69 | 250.047 | 2.51 | 1.8469 |
| 1.4 | 1.96 | 2.744 | 1.1882 | 1.1187 | 6.4 | 40.96 | 262.144 | 2.5298 | 1.8566 |
| 1.5 | 2.25 | 3.375 | 1.2247 | 1.1447 | 6.5 | 42.25 | 274.625 | 2.5495 | 1.8663 |
| 1.6 | 2.56 | 4.096 | 1.2649 | 1.1696 | 6.6 | 43.56 | 287.496 | 2.569 | 1.8758 |
| 1.7 | 2.89 | 4.913 | 1.3038 | 1.1935 | 6.7 | 44.89 | 300.763 | 2.5884 | 1.8852 |
| 1.8 | 3.24 | 5.832 | 1.3416 | 1.2164 | 6.8 | 46.24 | 314.432 | 2.6077 | 1.8945 |
| 1.9 | 3.61 | 6.859 | 1.3784 | 1.2386 | 6.9 | 47.61 | 328.509 | 2.6268 | 1.9038 |
| 2 | 4 | 8 | 1.4142 | 1.2599 | 7 | 49 | 343 | 2.6458 | 1.9129 |
| 2.1 | 4.41 | 9.261 | 1.4491 | 1.2806 | 7.1 | 50.41 | 357.911 | 2.6646 | 1.922 |
| 2.2 | 4.84 | 10.648 | 1.4832 | 1.3006 | 7.2 | 51.84 | 373.248 | 2.6833 | 1.931 |
| 2.3 | 5.29 | 12.167 | 1.5166 | 1.32 | 7.3 | 53.29 | 389.017 | 2.7019 | 1.9399 |
| 2.4 | 5.76 | 13.824 | 1.5492 | 1.3389 | 7.4 | 54.76 | 405.224 | 2.7203 | 1.9487 |
| 2.5 | 6.25 | 15.625 | 1.5811 | 1.3572 | 7.5 | 56.25 | 421.875 | 2.7386 | 1.9574 |
| 2.6 | 6.76 | 17.576 | 1.6125 | 1.3751 | 7.6 | 57.76 | 438.976 | 2.7568 | 1.9661 |
| 2.7 | 7.29 | 19.683 | 1.6432 | 1.3925 | 7.7 | 59.29 | 456.533 | 2.7749 | 1.9747 |
| 2.8 | 7.84 | 21.952 | 1.6733 | 1.4095 | 7.8 | 60.84 | 474.552 | 2.7928 | 1.9832 |
| 2.9 | 8.41 | 24.389 | 1.7029 | 1.426 | 7.9 | 62.41 | 493.039 | 2.8107 | 1.9916 |
| 3 | 9 | 27 | 1.7321 | 1.4422 | 8 | 64 | 512 | 2.8284 | 2 |
| 3.1 | 9.61 | 29.791 | 1.7607 | 1.4581 | 8.1 | 65.61 | 531.441 | 2.846 | 2.0083 |
| 3.2 | 10.24 | 32.768 | 1.7889 | 1.4736 | 8.2 | 67.24 | 551.868 | 2.8636 | 2.0165 |
| 3.3 | 10.89 | 35.937 | 1.8166 | 1.4889 | 8.3 | 68.89 | 573.787 | 2.881 | 2.0247 |
| 3.4 | 11.56 | 39.304 | 1.8439 | 1.5037 | 8.4 | 70.56 | 597.204 | 2.8983 | 2.0328 |
| 3.5 | 12.25 | 42.875 | 1.8708 | 1.5183 | 8.5 | 72.25 | 622.125 | 2.9155 | 2.0408 |
| 3.6 | 12.96 | 46.656 | 1.8971 | 1.5326 | 8.6 | 73.96 | 648.056 | 2.9326 | 2.0488 |
| 3.7 | 13.69 | 50.653 | 1.9235 | 1.5467 | 8.7 | 75.69 | 684.503 | 2.9496 | 2.0567 |
| 3.8 | 14.44 | 54.872 | 1.9494 | 1.5605 | 8.8 | 77.44 | 722.472 | 2.9665 | 2.0646 |
| 3.9 | 15.21 | 59.319 | 1.9748 | 1.5741 | 8.9 | 79.21 | 761.969 | 2.9833 | 2.0724 |
| 4 | 16 | 64 | 1.6 | 1.5871 | 9 | 81 | 729 | 3 | 2.0801 |
| 4.1 | 16.81 | 68.921 | 2.0249 | 1.6005 | 9.1 | 82.81 | 753.571 | 3.0166 | 2.0878 |
| 4.2 | 17.64 | 74.088 | 2.0494 | 1.6134 | 9.2 | 84.64 | 778.688 | 3.0332 | 2.0954 |
| 4.3 | 18.49 | 79.507 | 2.0736 | 1.6261 | 9.3 | 86.49 | 804.357 | 3.0496 | 2.1029 |
| 4.4 | 19.36 | 85.184 | 2.0976 | 1.6386 | 9.4 | 88.36 | 830.584 | 3.0659 | 2.1105 |
| 4.5 | 20.25 | 91.125 | 2.1213 | 1.651 | 9.5 | 90.25 | 857.375 | 3.0822 | 2.1179 |
| 4.6 | 21.16 | 97.336 | 2.1448 | 1.6631 | 9.6 | 92.16 | 884.736 | 3.0984 | 2.1253 |
| 4.7 | 22.09 | 103.823 | 2.1680 | 1.6751 | 9.7 | 94.09 | 912.673 | 3.1145 | 2.1327 |
| 4.8 | 23.04 | 110.692 | 2.1909 | 1.6869 | 9.8 | 96.04 | 941.192 | 3.1305 | 2.14 |
| 4.9 | 24.01 | 117.849 | 2.2136 | 1.6985 | 9.9 | 98.01 | 970.299 | 3.1464 | 2.1472 |
| 5 | 25 | 125 | 2.2361 | 1.71 | 10 | 100 | 1000 | 3.1623 | 2.1544 |
| 5.1 | 26.01 | 132.651 | 2.2583 | 1.7213 | 10.1 | 102.01 | 1030.301 | 3.178 | 2.1616 |
| 5.2 | 27.04 | 140.608 | 2.2804 | 1.7325 | 10.2 | 104.04 | 1061.208 | 3.1937 | 2.1687 |
| 5.3 | 28.09 | 148.977 | 2.3022 | 1.7435 | 10.3 | 106.09 | 1092.727 | 3.2094 | 2.1757 |
| 5.4 | 29.16 | 157.464 | 2.3238 | 1.7544 | 10.4 | 108.16 | 1124.868 | 3.2249 | 2.1828 |
| 5.5 | 30.25 | 166.375 | 2.3452 | 1.7652 | 10.5 | 110.25 | 1157.625 | 3.2404 | 2.1897 |
| 5.6 | 31.36 | 175.616 | 2.3664 | 1.7758 | 10.6 | 112.36 | 1191.016 | 3.2558 | 2.1967 |
| 5.7 | 32.49 | 185.193 | 2.3875 | 1.7863 | 10.7 | 114.49 | 1225.043 | 3.2711 | 2.2036 |
| 5.8 | 33.64 | 195.112 | 2.4083 | 1.7967 | 10.8 | 116.64 | 1259.712 | 3.2863 | 2.2104 |
| 5.9 | 34.81 | 205.379 | 2.429 | 1.807 | 10.9 | 118.81 | 1295.029 | 3.3015 | 2.2172 |

TABLES

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TABLE OF SQUARES, ETC.—Continued.

| No. | Square. | Cube. | Square Root. | Cube Root. | No. | Square. | Cube. | Square Root. | Cube Root. |
|------|---------|----------|--------------|------------|------|---------|----------|--------------|------------|
| 11 | 121 | 1331 | 3-3166 | 2-2389 | 16 | 256 | 4096 | 4 | 2-5198 |
| 11-1 | 128-21 | 1367-631 | 3-3317 | 2-2807 | 16-1 | 259-21 | 4178-281 | 4-0125 | 2-5251 |
| 11-2 | 125-44 | 1406-928 | 3-3466 | 2-2874 | 16-2 | 262-44 | 4251-528 | 4-0249 | 2-5308 |
| 11-3 | 127-69 | 1442-897 | 3-3615 | 2-2441 | 16-3 | 265-69 | 4380-747 | 4-0373 | 2-5355 |
| 11-4 | 129-96 | 1481-544 | 3-3764 | 2-2506 | 16-4 | 268-96 | 4410-944 | 4-0497 | 2-5407 |
| 11-5 | 132-25 | 1520-875 | 3-3912 | 2-2572 | 16-5 | 272-25 | 4492-125 | 4-062 | 2-5458 |
| 11-6 | 134-56 | 1560-896 | 3-4059 | 2-2637 | 16-6 | 275-56 | 4574-296 | 4-0743 | 2-5509 |
| 11-7 | 136-89 | 1601-613 | 3-4205 | 2-2702 | 16-7 | 278-89 | 4657-463 | 4-0866 | 2-5561 |
| 11-8 | 139-24 | 1643-032 | 3-4351 | 2-2766 | 16-8 | 282-24 | 4741-632 | 4-0988 | 2-5612 |
| 11-9 | 141-61 | 1685-159 | 3-4496 | 2-2831 | 16-9 | 285-61 | 4826-809 | 4-111 | 2-5662 |
| 12 | 144 | 1728 | 3-4641 | 2-2894 | 17 | 289 | 4913 | 4-1231 | 2-5713 |
| 12-1 | 146-41 | 1771-561 | 3-4785 | 2-2957 | 17-1 | 292-41 | 5000-211 | 4-1352 | 2-5763 |
| 12-2 | 148-84 | 1815-848 | 3-4928 | 2-3021 | 17-2 | 295-84 | 5088-448 | 4-1473 | 2-5813 |
| 12-3 | 151-29 | 1860-867 | 3-5071 | 2-3084 | 17-3 | 299-29 | 5177-717 | 4-1593 | 2-5863 |
| 12-4 | 153-76 | 1906-624 | 3-5214 | 2-3146 | 17-4 | 302-76 | 5268-024 | 4-1713 | 2-5913 |
| 12-5 | 156-25 | 1953-125 | 3-5355 | 2-3298 | 17-5 | 306-25 | 5359-375 | 4-1833 | 2-5962 |
| 12-6 | 158-76 | 2000-876 | 3-5496 | 2-327 | 17-6 | 309-76 | 5451-776 | 4-1952 | 2-6012 |
| 12-7 | 161-29 | 2048-883 | 3-5637 | 2-3331 | 17-7 | 313-29 | 5545-233 | 4-2071 | 2-6061 |
| 12-8 | 163-84 | 2097-152 | 3-5777 | 2-3391 | 17-8 | 316-84 | 5639-752 | 4-219 | 2-611 |
| 12-9 | 166-41 | 2146-689 | 3-5917 | 2-3453 | 17-9 | 320-41 | 5735-339 | 4-2308 | 2-6159 |
| 13 | 169 | 2197 | 3-6056 | 2-3513 | 18 | 324 | 5832 | 4-2426 | 2-6207 |
| 13-1 | 171-61 | 2248-091 | 3-6194 | 2-3573 | 18-1 | 327-61 | 5929-741 | 4-2544 | 2-6256 |
| 13-2 | 174-24 | 2299-968 | 3-6332 | 2-3633 | 18-2 | 331-24 | 6028-568 | 4-2661 | 2-6304 |
| 13-3 | 176-89 | 2352-637 | 3-6469 | 2-3693 | 18-3 | 334-89 | 6128-487 | 4-2778 | 2-6352 |
| 13-4 | 179-56 | 2406-104 | 3-6606 | 2-3752 | 18-4 | 338-56 | 6229-504 | 4-2895 | 2-64 |
| 13-5 | 182-25 | 2460-375 | 3-6742 | 2-3811 | 18-5 | 342-25 | 6331-625 | 4-3012 | 2-6448 |
| 13-6 | 184-96 | 2515-456 | 3-6878 | 2-387 | 18-6 | 345-96 | 6434-856 | 4-3128 | 2-6495 |
| 13-7 | 187-69 | 2571-353 | 3-7013 | 2-3928 | 18-7 | 349-69 | 6539-203 | 4-3248 | 2-6543 |
| 13-8 | 190-44 | 2628-072 | 3-7148 | 2-3986 | 18-8 | 353-44 | 6644-672 | 4-3359 | 2-659 |
| 13-9 | 193-21 | 2685-619 | 3-7283 | 2-4044 | 18-9 | 357-21 | 6751-269 | 4-3474 | 2-6637 |
| 14 | 196 | 2744 | 3-7417 | 2-4101 | 19 | 361 | 6859 | 4-3589 | 2-6684 |
| 14-1 | 198-81 | 2803-221 | 3-755 | 2-4159 | 19-1 | 364-81 | 6967-871 | 4-3704 | 2-6731 |
| 14-2 | 201-64 | 2863-284 | 3-7683 | 2-4216 | 19-2 | 368-64 | 7077-888 | 4-3818 | 2-6777 |
| 14-3 | 204-49 | 2924-207 | 3-7815 | 2-4272 | 19-3 | 372-49 | 7189-057 | 4-3932 | 2-6824 |
| 14-4 | 207-36 | 2985-984 | 3-7947 | 2-4329 | 19-4 | 376-36 | 7301-381 | 4-4045 | 2-6870 |
| 14-5 | 210-26 | 3048-625 | 3-8079 | 2-4385 | 19-5 | 380-25 | 7414-875 | 4-4159 | 2-6916 |
| 14-6 | 213-16 | 3112-136 | 3-821 | 2-4441 | 19-6 | 384-16 | 7529-536 | 4-4272 | 2-6962 |
| 14-7 | 216-09 | 3176-523 | 3-8341 | 2-4497 | 19-7 | 388-09 | 7645-373 | 4-4385 | 2-7008 |
| 14-8 | 219-04 | 3241-792 | 3-8471 | 2-4552 | 19-8 | 392-04 | 7762-392 | 4-4497 | 2-7053 |
| 14-9 | 222-01 | 3307-949 | 3-86 | 2-4607 | 19-9 | 396-01 | 7880-599 | 4-4609 | 2-7099 |
| 15 | 225 | 3375 | 3-873 | 2-4662 | 20 | 400 | 8000 | 4-4721 | 2-7144 |
| 15-1 | 228-01 | 3442-951 | 3-8859 | 2-4717 | 20-1 | 404-01 | 8120-601 | 4-4833 | 2-7189 |
| 15-2 | 231-04 | 3511-808 | 3-8987 | 2-4771 | 20-2 | 408-04 | 8242-408 | 4-4944 | 2-7234 |
| 15-3 | 234-09 | 3581-577 | 3-9115 | 2-4825 | 20-3 | 412-09 | 8365-427 | 4-5055 | 2-7279 |
| 15-4 | 237-16 | 3652-264 | 3-9243 | 2-4879 | 20-4 | 416-16 | 8489-664 | 4-5166 | 2-7324 |
| 15-5 | 240-25 | 3723-875 | 3-937 | 2-4933 | 20-5 | 420-25 | 8615-125 | 4-5277 | 2-7368 |
| 15-6 | 243-36 | 3796-416 | 3-9497 | 2-4987 | 20-6 | 424-36 | 8741-816 | 4-5387 | 2-7413 |
| 15-7 | 246-49 | 3869-898 | 3-9623 | 2-504 | 20-7 | 428-49 | 8869-748 | 4-5497 | 2-7457 |
| 15-8 | 249-64 | 3944-812 | 3-9749 | 2-5098 | 20-8 | 432-64 | 8998-912 | 4-5607 | 2-7502 |
| 15-9 | 252-81 | 4019-679 | 3-9875 | 2-5146 | 20-9 | 436-81 | 9129-829 | 4-5716 | 2-7545 |

TABLE OF SQUARES, ETC.—Continued.

| No. | Square. | Cube. | Square Root. | Cube Root. | No. | Square. | Cube. | Square Root. | Cube Root. |
|------|---------|-----------|--------------|------------|------|---------|-----------|--------------|------------|
| 21 | 441 | 9261 | 4.5826 | 2.7539 | 26 | 676 | 17576 | 5.099 | 2.9625 |
| 21.1 | 445.21 | 9398.931 | 4.5935 | 2.7633 | 26.1 | 681.21 | 17779.581 | 5.1088 | 2.9668 |
| 21.2 | 449.44 | 9528.128 | 4.6043 | 2.7676 | 26.2 | 686.44 | 17984.728 | 5.1186 | 2.9701 |
| 21.8 | 453.69 | 9668.597 | 4.6152 | 2.772 | 26.3 | 691.69 | 18191.447 | 5.1284 | 2.9738 |
| 21.4 | 457.96 | 9800.844 | 4.626 | 2.7763 | 26.4 | 696.96 | 18399.744 | 5.1381 | 2.9776 |
| 21.5 | 462.25 | 9938.875 | 4.6368 | 2.7806 | 26.5 | 702.25 | 18609.625 | 5.1478 | 2.9814 |
| 21.6 | 466.56 | 10077.696 | 4.6476 | 2.7849 | 26.6 | 707.56 | 18821.096 | 5.1575 | 2.9851 |
| 21.7 | 470.89 | 10218.313 | 4.6586 | 2.7893 | 26.7 | 712.89 | 19034.163 | 5.1672 | 2.9888 |
| 21.8 | 475.24 | 10360.232 | 4.669 | 2.7935 | 26.8 | 718.24 | 19248.832 | 5.1769 | 2.9926 |
| 21.9 | 479.61 | 10508.459 | 4.6797 | 2.7978 | 26.9 | 723.61 | 19465.109 | 5.1865 | 2.9963 |
| 22 | 484 | 10648 | 4.6904 | 2.8021 | 27 | 729 | 19683 | 5.1962 | 3 |
| 22.1 | 488.41 | 10793.861 | 4.7011 | 2.8063 | 27.1 | 734.41 | 19902.511 | 5.2058 | 3.0037 |
| 22.2 | 492.84 | 10941.048 | 4.7117 | 2.8105 | 27.2 | 739.84 | 20123.648 | 5.2154 | 3.0074 |
| 22.3 | 497.29 | 11089.567 | 4.7223 | 2.8147 | 27.3 | 745.29 | 20346.417 | 5.2249 | 3.0111 |
| 22.4 | 501.76 | 11239.424 | 4.7329 | 2.8189 | 27.4 | 750.76 | 20570.824 | 5.2345 | 3.0147 |
| 22.5 | 506.25 | 11390.625 | 4.7434 | 2.8231 | 27.5 | 756.25 | 20796.875 | 5.244 | 3.0184 |
| 22.6 | 510.76 | 11543.176 | 4.7539 | 2.8273 | 27.6 | 761.76 | 21024.576 | 5.2536 | 3.0221 |
| 22.7 | 515.29 | 11697.083 | 4.7644 | 2.8314 | 27.7 | 767.29 | 21253.933 | 5.2631 | 3.0257 |
| 22.8 | 519.84 | 11852.352 | 4.7749 | 2.8356 | 27.8 | 772.84 | 21484.952 | 5.2726 | 3.0293 |
| 22.9 | 524.41 | 12008.989 | 4.7854 | 2.8397 | 27.9 | 778.41 | 21717.639 | 5.282 | 3.033 |
| 23 | 529 | 12167 | 4.7958 | 2.8438 | 28 | 784 | 21952 | 5.2915 | 3.0366 |
| 23.1 | 533.61 | 12326.391 | 4.8062 | 2.8479 | 28.1 | 789.61 | 22188.041 | 5.3009 | 3.0402 |
| 23.2 | 538.24 | 12487.168 | 4.8166 | 2.8521 | 28.2 | 795.24 | 22425.768 | 5.3104 | 3.0438 |
| 23.3 | 542.89 | 12649.337 | 4.827 | 2.8562 | 28.3 | 800.89 | 22665.187 | 5.3198 | 3.0474 |
| 23.4 | 547.56 | 12812.904 | 4.8373 | 2.8603 | 28.4 | 806.56 | 22906.304 | 5.3292 | 3.051 |
| 23.5 | 552.25 | 12977.875 | 4.8477 | 2.8643 | 28.5 | 812.25 | 23149.125 | 5.3385 | 3.0546 |
| 23.6 | 556.96 | 13144.266 | 4.858 | 2.8684 | 28.6 | 817.96 | 23393.656 | 5.3479 | 3.0581 |
| 23.7 | 561.69 | 13312.053 | 4.8683 | 2.8724 | 28.7 | 823.69 | 23639.903 | 5.3572 | 3.0617 |
| 23.8 | 566.44 | 13481.272 | 4.8785 | 2.8765 | 28.8 | 829.44 | 23887.872 | 5.3666 | 3.0652 |
| 23.9 | 571.21 | 13651.919 | 4.8888 | 2.8805 | 28.9 | 835.21 | 24137.569 | 5.3759 | 3.0688 |
| 24 | 576 | 13824 | 4.899 | 2.8845 | 29 | 841 | 24389 | 5.3852 | 3.0723 |
| 24.1 | 580.81 | 13997.521 | 4.9092 | 2.8885 | 29.1 | 846.81 | 24642.171 | 5.3944 | 3.0758 |
| 24.2 | 585.64 | 14172.488 | 4.9193 | 2.8925 | 29.2 | 852.64 | 24897.088 | 5.4037 | 3.0794 |
| 24.3 | 590.49 | 14348.90 | 4.9295 | 2.8965 | 29.3 | 858.49 | 25153.757 | 5.4129 | 3.0829 |
| 24.4 | 595.36 | 14526.784 | 4.9396 | 2.9004 | 29.4 | 864.36 | 25412.184 | 5.4222 | 3.0864 |
| 24.5 | 600.25 | 14706.125 | 4.9497 | 2.9044 | 29.5 | 870.25 | 25672.375 | 5.4314 | 3.0899 |
| 24.6 | 605.16 | 14886.936 | 4.9598 | 2.9083 | 29.6 | 876.16 | 25934.336 | 5.4406 | 3.0934 |
| 24.7 | 610.09 | 15069.223 | 4.9699 | 2.9123 | 29.7 | 882.09 | 26198.073 | 5.4498 | 3.0968 |
| 24.8 | 615.04 | 15252.992 | 4.9799 | 2.9162 | 29.8 | 888.04 | 26463.592 | 5.4589 | 3.1003 |
| 24.9 | 620.01 | 15438.249 | 4.9899 | 2.9201 | 29.9 | 894.01 | 26730.899 | 5.4681 | 3.1038 |
| 25 | 625 | 15625 | 5 | 2.9240 | 30 | 900 | 27000 | 5.4772 | 3.1072 |
| 25.1 | 630.01 | 15813.251 | 5.01 | 2.9279 | 30.1 | 906.01 | 27270.901 | 5.4863 | 3.1107 |
| 25.2 | 635.04 | 16003.008 | 5.02 | 2.9318 | 30.2 | 912.04 | 27543.608 | 5.4954 | 3.1141 |
| 25.3 | 640.09 | 16194.277 | 5.0299 | 2.9357 | 30.3 | 918.09 | 27818.127 | 5.5045 | 3.1176 |
| 25.4 | 645.16 | 16387.064 | 5.0398 | 2.9395 | 30.4 | 924.16 | 28094.464 | 5.5136 | 3.121 |
| 25.5 | 650.25 | 16581.375 | 5.0498 | 2.9434 | 30.5 | 930.25 | 28372.625 | 5.5226 | 3.1244 |
| 25.6 | 655.36 | 16777.216 | 5.0596 | 2.9472 | 30.6 | 936.36 | 28652.616 | 5.5317 | 3.1278 |
| 25.7 | 660.49 | 16974.593 | 5.0695 | 2.9511 | 30.7 | 942.49 | 28934.443 | 5.5407 | 3.1312 |
| 25.8 | 665.64 | 17173.512 | 5.0794 | 2.9549 | 30.8 | 948.64 | 29218.112 | 5.5497 | 3.1346 |
| 25.9 | 670.81 | 17373.979 | 5.0892 | 2.9587 | 30.9 | 954.81 | 29508.629 | 5.5587 | 3.138 |

TABLE OF SQUARES, ETC.—Continued.

| No. | Square. | Cube. | Square Root. | Cube Root. | No. | Square. | Cube. | Square Root. | Cube Root. |
|------|---------|-----------|--------------|------------|------|---------|-----------|--------------|------------|
| 31 | 961 | 29791 | 5-5678 | 3-2414 | 36 | 1296 | 46656 | 6 | 3-3019 |
| 31-1 | 967-21 | 30080-231 | 5-5767 | 3-1448 | 36-1 | 1308-21 | 47045-881 | 6-0083 | 3-306 |
| 31-2 | 973-44 | 30371-328 | 5-5857 | 3-1481 | 36-2 | 1310-44 | 47437-928 | 6-0166 | 3-308 |
| 31-3 | 979-69 | 30664-297 | 5-5946 | 3-1515 | 36-3 | 1317-69 | 47832-147 | 6-0249 | 3-3111 |
| 31-4 | 985-96 | 30959-144 | 5-6035 | 3-1548 | 36-4 | 1324-96 | 48228-544 | 6-0332 | 3-3141 |
| 31-5 | 992-25 | 31255-875 | 5-6124 | 3-1582 | 36-5 | 1332-25 | 48627-125 | 6-0415 | 3-3171 |
| 31-6 | 998-56 | 31554-496 | 5-6213 | 3-1615 | 36-6 | 1339-56 | 49027-896 | 6-0498 | 3-3202 |
| 31-7 | 1004-89 | 31855-013 | 5-6302 | 3-1648 | 36-7 | 1346-89 | 49430-808 | 6-0581 | 3-3232 |
| 31-8 | 1011-24 | 32157-432 | 5-6391 | 3-1674 | 36-8 | 1354-24 | 49836-032 | 6-0663 | 3-3262 |
| 31-9 | 1017-61 | 32461-759 | 5-648 | 3-1715 | 36-9 | 1361-61 | 50243-409 | 6-0745 | 3-3292 |
| 32 | 1024 | 32768 | 5-6569 | 3-1748 | 37 | 1369 | 50653 | 6-0828 | 3-3322 |
| 32-1 | 1030-41 | 33076-161 | 5-6656 | 3-1781 | 37-1 | 1376-41 | 51064-811 | 6-091 | 3-3352 |
| 32-2 | 1036-84 | 33386-248 | 5-6745 | 3-1814 | 37-2 | 1383-84 | 51478-848 | 6-0992 | 3-3382 |
| 32-3 | 1043-29 | 33698-267 | 5-6833 | 3-1847 | 37-3 | 1391-29 | 51895-117 | 6-1074 | 3-3412 |
| 32-4 | 1049-76 | 34012-224 | 5-6921 | 3-188 | 37-4 | 1398-76 | 52313-624 | 6-1156 | 3-3442 |
| 32-5 | 1056-25 | 34328-125 | 5-7008 | 3-1913 | 37-5 | 1406-25 | 52734-375 | 6-1237 | 3-3472 |
| 32-6 | 1062-76 | 34645-976 | 5-7096 | 3-1945 | 37-6 | 1413-76 | 53157-876 | 6-1319 | 3-3501 |
| 32-7 | 1069-29 | 34965-783 | 5-7183 | 3-1978 | 37-7 | 1421-29 | 53582-633 | 6-14 | 3-3531 |
| 32-8 | 1075-84 | 35287-552 | 5-7271 | 3-201 | 37-8 | 1428-84 | 54010-152 | 6-1482 | 3-3561 |
| 32-9 | 1082-41 | 35611-289 | 5-7358 | 3-2043 | 37-9 | 1436-41 | 54439-939 | 6-1563 | 3-359 |
| 33 | 1089 | 35937 | 5-7446 | 3-2075 | 38 | 1444 | 54872 | 6-1644 | 3-362 |
| 33-1 | 1095-61 | 36264-691 | 5-7532 | 3-2108 | 38-1 | 1451-61 | 55306-341 | 6-1725 | 3-3649 |
| 33-2 | 1102-24 | 36594-368 | 5-7619 | 3-214 | 38-2 | 1459-24 | 55742-968 | 6-1806 | 3-3679 |
| 33-3 | 1108-89 | 36926-037 | 5-7706 | 3-2172 | 38-3 | 1466-89 | 56181-887 | 6-1887 | 3-3708 |
| 33-4 | 1115-56 | 37259-704 | 5-7792 | 3-2204 | 38-4 | 1474-56 | 56623-104 | 6-1968 | 3-3737 |
| 33-5 | 1122-25 | 37595-375 | 5-7879 | 3-2237 | 38-5 | 1482-25 | 57066-625 | 6-2048 | 3-3767 |
| 33-6 | 1128-96 | 37933-056 | 5-7965 | 3-2269 | 38-6 | 1489-96 | 57512-456 | 6-2129 | 3-3796 |
| 33-7 | 1135-69 | 38272-753 | 5-8051 | 3-2301 | 38-7 | 1497-69 | 57960-603 | 6-2209 | 3-3825 |
| 33-8 | 1142-44 | 38614-472 | 5-8137 | 3-2332 | 38-8 | 1505-44 | 58411-072 | 6-229 | 3-3854 |
| 33-9 | 1149-21 | 38958-211 | 5-8223 | 3-2364 | 38-9 | 1513-21 | 58863-869 | 6-237 | 3-3883 |
| 34 | 1156 | 39304 | 5-831 | 3-2396 | 39 | 1521 | 59319 | 6-245 | 3-3912 |
| 34-1 | 1162-81 | 39651-821 | 5-8395 | 3-2428 | 39-1 | 1528-81 | 59776-471 | 6-253 | 3-3941 |
| 34-2 | 1169-64 | 40002-688 | 5-848 | 3-246 | 39-2 | 1536-64 | 60236-288 | 6-261 | 3-397 |
| 34-3 | 1176-49 | 40353-607 | 5-8566 | 3-2491 | 39-3 | 1544-49 | 60698-457 | 6-269 | 3-3999 |
| 34-4 | 1183-36 | 40707-584 | 5-8651 | 3-2522 | 39-4 | 1552-36 | 61162-984 | 6-2769 | 3-4028 |
| 34-5 | 1190-25 | 41063-625 | 5-8736 | 3-2554 | 39-5 | 1560-25 | 61629-875 | 6-2849 | 3-4056 |
| 34-6 | 1197-16 | 41421-736 | 5-8821 | 3-2586 | 39-6 | 1568-16 | 62099-136 | 6-2929 | 3-4085 |
| 34-7 | 1204-09 | 41781-923 | 5-8906 | 3-2617 | 39-7 | 1576-09 | 62570-773 | 6-3008 | 3-4114 |
| 34-8 | 1211-04 | 42144-192 | 5-8991 | 3-2648 | 39-8 | 1584-04 | 63044-792 | 6-3087 | 3-4142 |
| 34-9 | 1218-01 | 42508-549 | 5-9076 | 3-2679 | 39-9 | 1592-01 | 63521-199 | 6-3166 | 3-4171 |
| 35 | 1225 | 42875 | 5-9161 | 3-2711 | 40 | 1600 | 64000 | 6-3246 | 3-42 |
| 35-1 | 1232-01 | 43248-551 | 5-9245 | 3-2742 | 40-1 | 1608-01 | 64481-201 | 6-3325 | 3-4228 |
| 35-2 | 1239-04 | 43614-208 | 5-933 | 3-2773 | 40-2 | 1616-04 | 64964-808 | 6-3404 | 3-4256 |
| 35-3 | 1246-09 | 43986-977 | 5-9414 | 3-2804 | 40-3 | 1624-09 | 65450-827 | 6-3482 | 3-4285 |
| 35-4 | 1253-16 | 44361-864 | 5-9498 | 3-2835 | 40-4 | 1632-16 | 65939-264 | 6-3561 | 3-4313 |
| 35-5 | 1260-25 | 44738-875 | 5-9582 | 3-2866 | 40-5 | 1640-25 | 66436-125 | 6-3639 | 3-4341 |
| 35-6 | 1267-36 | 45118-016 | 5-9666 | 3-2897 | 40-6 | 1648-36 | 66923-416 | 6-3718 | 3-437 |
| 35-7 | 1274-49 | 45499-293 | 5-9749 | 3-2927 | 40-7 | 1656-49 | 67419-143 | 6-3796 | 3-4398 |
| 35-8 | 1281-64 | 45882-712 | 5-9833 | 3-2958 | 40-8 | 1664-64 | 67911-312 | 6-3875 | 3-4426 |
| 35-9 | 1288-81 | 46268-279 | 5-9917 | 3-2989 | 40-9 | 1672-81 | 68417-929 | 6-3953 | 3-4454 |

TABLE OF SQUARES, ETC.—Continued.

| No. | Square. | Cube. | Square Root. | Cube Root. | No. | Square. | Cube. | Square Root. | Cube Root. |
|------|---------|-----------|--------------|------------|------|---------|------------|--------------|------------|
| 41 | 1681 | 68921 | 4081 | 3.4482 | 46 | 2116 | 97336 | 45.923 | 3.583 |
| 41.1 | 1689.21 | 69426.531 | 41.09 | 3.451 | 46.1 | 2125.21 | 97972.181 | 46.079 | 3.586 |
| 41.2 | 1697.44 | 69934.528 | 41.187 | 3.4538 | 46.2 | 2134.44 | 98611.128 | 46.237 | 3.5882 |
| 41.3 | 1705.69 | 70444.597 | 41.285 | 3.4566 | 46.3 | 2143.69 | 99252.847 | 46.396 | 3.5908 |
| 41.4 | 1713.96 | 70957.944 | 41.383 | 3.4594 | 46.4 | 2152.96 | 99897.344 | 46.556 | 3.5934 |
| 41.5 | 1722.25 | 71473.375 | 41.481 | 3.4622 | 46.5 | 2162.25 | 100544.625 | 46.717 | 3.596 |
| 41.6 | 1730.56 | 71991.296 | 41.579 | 3.465 | 46.6 | 2171.56 | 101194.696 | 46.878 | 3.5986 |
| 41.7 | 1738.89 | 72511.713 | 41.677 | 3.4677 | 46.7 | 2180.89 | 101847.563 | 47.039 | 3.6011 |
| 41.8 | 1747.24 | 73034.632 | 41.775 | 3.4705 | 46.8 | 2190.24 | 102503.232 | 47.200 | 3.6037 |
| 41.9 | 1755.61 | 73560.059 | 41.873 | 3.4733 | 46.9 | 2199.61 | 103161.709 | 47.361 | 3.6063 |
| 42 | 1764 | 74088 | 41.971 | 3.476 | 47 | 2209 | 103823 | 47.522 | 3.6088 |
| 42.1 | 1772.41 | 74618.461 | 42.069 | 3.4788 | 47.1 | 2218.41 | 104487.111 | 47.683 | 3.6114 |
| 42.2 | 1780.84 | 75151.448 | 42.167 | 3.4815 | 47.2 | 2227.84 | 105154.048 | 47.844 | 3.6139 |
| 42.3 | 1789.29 | 75686.967 | 42.265 | 3.4843 | 47.3 | 2237.29 | 105823.817 | 48.005 | 3.6165 |
| 42.4 | 1797.76 | 76225.024 | 42.363 | 3.487 | 47.4 | 2246.76 | 106496.424 | 48.166 | 3.619 |
| 42.5 | 1806.25 | 76765.625 | 42.461 | 3.4898 | 47.5 | 2256.25 | 107171.875 | 48.327 | 3.6216 |
| 42.6 | 1814.76 | 77308.776 | 42.559 | 3.4925 | 47.6 | 2265.76 | 107850.176 | 48.488 | 3.6241 |
| 42.7 | 1823.29 | 77854.483 | 42.657 | 3.4952 | 47.7 | 2275.29 | 108531.333 | 48.649 | 3.6267 |
| 42.8 | 1831.84 | 78402.752 | 42.755 | 3.498 | 47.8 | 2284.84 | 109215.352 | 48.810 | 3.6292 |
| 42.9 | 1840.41 | 78953.589 | 42.853 | 3.5007 | 47.9 | 2294.41 | 109902.239 | 48.971 | 3.6317 |
| 43 | 1849 | 79507 | 42.951 | 3.5034 | 48 | 2304 | 110592 | 49.132 | 3.6342 |
| 43.1 | 1857.61 | 80062.991 | 43.049 | 3.5061 | 48.1 | 2313.61 | 111284.641 | 49.293 | 3.6368 |
| 43.2 | 1866.24 | 80621.568 | 43.147 | 3.5088 | 48.2 | 2323.24 | 111980.168 | 49.454 | 3.6393 |
| 43.3 | 1874.89 | 81182.737 | 43.245 | 3.5115 | 48.3 | 2332.89 | 112678.587 | 49.615 | 3.6418 |
| 43.4 | 1883.56 | 81746.504 | 43.343 | 3.5142 | 48.4 | 2342.56 | 113379.904 | 49.776 | 3.6443 |
| 43.5 | 1892.25 | 82312.875 | 43.441 | 3.5169 | 48.5 | 2352.25 | 114084.125 | 49.937 | 3.6468 |
| 43.6 | 1900.96 | 82881.856 | 43.539 | 3.5196 | 48.6 | 2361.96 | 114791.256 | 50.098 | 3.6493 |
| 43.7 | 1909.69 | 83453.453 | 43.637 | 3.5223 | 48.7 | 2371.69 | 115501.303 | 50.259 | 3.6513 |
| 43.8 | 1918.44 | 84027.672 | 43.735 | 3.525 | 48.8 | 2381.44 | 116214.272 | 50.420 | 3.6543 |
| 43.9 | 1927.21 | 84604.519 | 43.833 | 3.5277 | 48.9 | 2391.21 | 116930.169 | 50.581 | 3.6568 |
| 44 | 1936 | 85184 | 43.931 | 3.5303 | 49 | 2401 | 117649 | 50.742 | 3.6593 |
| 44.1 | 1944.81 | 85766.121 | 44.029 | 3.533 | 49.1 | 2410.81 | 118370.771 | 50.903 | 3.6618 |
| 44.2 | 1953.64 | 86350.888 | 44.127 | 3.5357 | 49.2 | 2420.64 | 119095.488 | 51.064 | 3.6643 |
| 44.3 | 1962.49 | 86938.307 | 44.225 | 3.5384 | 49.3 | 2430.49 | 119823.157 | 51.225 | 3.6668 |
| 44.4 | 1971.36 | 87528.384 | 44.323 | 3.541 | 49.4 | 2440.36 | 120553.784 | 51.386 | 3.6692 |
| 44.5 | 1980.25 | 88121.125 | 44.421 | 3.5437 | 49.5 | 2450.25 | 121287.375 | 51.547 | 3.6717 |
| 44.6 | 1989.16 | 88716.536 | 44.519 | 3.5463 | 49.6 | 2460.16 | 122023.936 | 51.708 | 3.6742 |
| 44.7 | 1998.09 | 89314.623 | 44.617 | 3.549 | 49.7 | 2470.09 | 122763.473 | 51.869 | 3.6766 |
| 44.8 | 2007.04 | 89915.392 | 44.715 | 3.5516 | 49.8 | 2480.04 | 123505.992 | 52.030 | 3.6791 |
| 44.9 | 2016.01 | 90518.849 | 44.813 | 3.5543 | 49.9 | 2490.01 | 124251.499 | 52.191 | 3.6816 |
| 45 | 2025 | 91125 | 44.911 | 3.5569 | 50 | 2500 | 125000 | 52.352 | 3.684 |
| 45.1 | 2034.01 | 91733.851 | 45.009 | 3.5595 | 50.1 | 2509.01 | 125751.001 | 52.513 | 3.6865 |
| 45.2 | 2043.04 | 92345.408 | 45.107 | 3.5622 | 50.2 | 2518.04 | 126504.04 | 52.674 | 3.689 |
| 45.3 | 2052.09 | 92959.677 | 45.205 | 3.5648 | 50.3 | 2527.09 | 127259.09 | 52.835 | 3.6915 |
| 45.4 | 2061.16 | 93576.664 | 45.303 | 3.5674 | 50.4 | 2536.16 | 128016.16 | 52.996 | 3.694 |
| 45.5 | 2070.25 | 94196.375 | 45.401 | 3.57 | 50.5 | 2545.25 | 128775.25 | 53.157 | 3.6965 |
| 45.6 | 2079.36 | 94818.816 | 45.499 | 3.5726 | 50.6 | 2554.36 | 129536.36 | 53.318 | 3.699 |
| 45.7 | 2088.49 | 95443.993 | 45.597 | 3.5752 | 50.7 | 2563.49 | 130299.49 | 53.479 | 3.7015 |
| 45.8 | 2097.64 | 96071.912 | 45.695 | 3.5778 | 50.8 | 2572.64 | 131064.64 | 53.640 | 3.704 |
| 45.9 | 2106.81 | 96702.579 | 45.793 | 3.5805 | 50.9 | 2581.81 | 131831.81 | 53.801 | 3.7065 |

TABLE OF SQUARES, ETC.—Continued.

| No. | Square. | Cube. | Square Root. | Cube Root. | No. | Square. | Cube. | Square Root. | Cube Root. |
|-----|---------|--------|--------------|------------|-----|---------|--------|--------------|------------|
| 60 | 3600 | 216000 | 7.746 | 3.9149 | 80 | 6400 | 512000 | 8.9443 | 4.3089 |
| 61 | 3721 | 226981 | 7.8102 | 3.9365 | 81 | 6561 | 531441 | 9 | 4.3267 |
| 62 | 3844 | 238328 | 7.874 | 3.9579 | 82 | 6724 | 551368 | 9.0554 | 4.3445 |
| 63 | 3969 | 250047 | 7.9373 | 3.9791 | 83 | 6889 | 571787 | 9.1104 | 4.3621 |
| 64 | 4096 | 262144 | 8 | 4 | 84 | 7056 | 592704 | 9.1652 | 4.3795 |
| 65 | 4225 | 274625 | 8.0623 | 4.0207 | 85 | 7225 | 614125 | 9.2195 | 4.3969 |
| 66 | 4356 | 287496 | 8.124 | 4.0412 | 86 | 7396 | 636056 | 9.2736 | 4.414 |
| 67 | 4489 | 300763 | 8.1854 | 4.0615 | 87 | 7569 | 658508 | 9.3274 | 4.431 |
| 68 | 4624 | 314432 | 8.2462 | 4.0817 | 88 | 7744 | 681472 | 9.3808 | 4.448 |
| 69 | 4761 | 328509 | 8.3066 | 4.1016 | 89 | 7921 | 704969 | 9.434 | 4.4647 |
| 70 | 4900 | 343000 | 8.3666 | 4.1213 | 90 | 8100 | 729000 | 9.4868 | 4.4814 |
| 71 | 5041 | 357911 | 8.4261 | 4.1408 | 91 | 8281 | 753571 | 9.5394 | 4.4979 |
| 72 | 5184 | 373248 | 8.4853 | 4.1602 | 92 | 8464 | 778688 | 9.5917 | 4.5144 |
| 73 | 5329 | 389017 | 8.544 | 4.1793 | 93 | 8649 | 804357 | 9.6437 | 4.5307 |
| 74 | 5476 | 405224 | 8.6023 | 4.1983 | 94 | 8836 | 830584 | 9.6954 | 4.5468 |
| 75 | 5625 | 421875 | 8.6603 | 4.2172 | 95 | 9025 | 857375 | 9.7468 | 4.5629 |
| 76 | 5776 | 438976 | 8.7178 | 4.2358 | 96 | 9216 | 884736 | 9.798 | 4.5789 |
| 77 | 5929 | 456533 | 8.775 | 4.2543 | 97 | 9409 | 912673 | 9.8489 | 4.5947 |
| 78 | 6084 | 474552 | 8.8318 | 4.2727 | 98 | 9604 | 941192 | 9.8995 | 4.6104 |
| 79 | 6241 | 493039 | 8.8882 | 4.2908 | 99 | 9801 | 970299 | 9.9499 | 4.6261 |

AREAS AND CIRCUMFERENCES OF CIRCLES UP TO 6 IN. DIAMETER
(ADVANCING BY 32NDS AND 16THS)

| Dia. | Circum. | Area. | Dia. | Circum. | Area. | Dia. | Circum. | Area. |
|------------------|---------|---------|-----------------|---------|--------|-----------------|---------|--------|
| $\frac{1}{32}$ | 0.0981 | 0.00077 | $1\frac{1}{32}$ | 3.3197 | 1.4848 | $3\frac{1}{32}$ | 11.584 | 10.674 |
| $\frac{1}{16}$ | .1963 | .00307 | $1\frac{1}{16}$ | 4.516 | 1.6229 | $3\frac{1}{16}$ | 11.791 | 11.044 |
| $\frac{3}{32}$ | .2945 | .0073 | $1\frac{3}{32}$ | 4.7124 | 1.7671 | $3\frac{3}{32}$ | 11.977 | 11.416 |
| $\frac{1}{8}$ | .3927 | .01227 | $1\frac{1}{8}$ | 4.9087 | 1.9175 | $3\frac{1}{8}$ | 12.173 | 11.793 |
| $\frac{5}{32}$ | .4908 | .0192 | $1\frac{5}{32}$ | 5.1051 | 2.0739 | $3\frac{5}{32}$ | 12.369 | 12.177 |
| $\frac{3}{16}$ | .589 | .02761 | $1\frac{3}{16}$ | 5.3014 | 2.2365 | 4 | 12.566 | 12.566 |
| $\frac{7}{32}$ | .6872 | .0376 | $1\frac{7}{32}$ | 5.4978 | 2.4052 | $4\frac{1}{32}$ | 12.762 | 12.962 |
| $\frac{1}{4}$ | .7854 | .04909 | $1\frac{1}{4}$ | 5.6941 | 2.58 | $4\frac{1}{4}$ | 12.959 | 13.364 |
| $\frac{9}{32}$ | .8835 | .0621 | $1\frac{9}{32}$ | 5.8905 | 2.7611 | $4\frac{1}{8}$ | 13.155 | 13.772 |
| $\frac{5}{16}$ | .9817 | .0767 | $1\frac{5}{16}$ | 6.0868 | 2.9483 | $4\frac{1}{2}$ | 13.351 | 14.186 |
| $\frac{3}{8}$ | 1.0799 | .0928 | 2 | 6.2832 | 3.1416 | $4\frac{3}{8}$ | 13.547 | 14.606 |
| $\frac{7}{16}$ | 1.1781 | .1104 | $2\frac{1}{16}$ | 6.4795 | 3.3410 | $4\frac{7}{16}$ | 13.744 | 15.038 |
| $\frac{1}{2}$ | 1.2762 | .1296 | $2\frac{1}{8}$ | 6.6759 | 3.5465 | $4\frac{1}{2}$ | 13.94 | 15.465 |
| $\frac{5}{8}$ | 1.3744 | .1503 | $2\frac{3}{8}$ | 6.8722 | 3.7584 | $4\frac{5}{8}$ | 14.137 | 15.904 |
| $\frac{3}{4}$ | 1.4726 | .1725 | $2\frac{1}{2}$ | 7.0686 | 3.976 | $4\frac{3}{4}$ | 14.333 | 16.349 |
| $\frac{7}{8}$ | 1.5708 | .1963 | $2\frac{5}{8}$ | 7.2649 | 4.2 | $4\frac{7}{8}$ | 14.529 | 16.8 |
| $1\frac{1}{32}$ | 1.6689 | .2216 | $2\frac{3}{4}$ | 7.4613 | 4.4302 | $4\frac{1}{2}$ | 14.725 | 17.257 |
| $1\frac{1}{16}$ | 1.7671 | .2485 | $2\frac{7}{8}$ | 7.6576 | 4.6664 | $4\frac{1}{2}$ | 14.922 | 17.72 |
| $1\frac{3}{32}$ | 1.8653 | .2768 | $2\frac{1}{2}$ | 7.854 | 4.9087 | $4\frac{1}{2}$ | 15.119 | 18.19 |
| $1\frac{1}{8}$ | 1.9635 | .3068 | $2\frac{5}{8}$ | 8.0503 | 5.1572 | $4\frac{1}{2}$ | 15.315 | 18.665 |
| $1\frac{3}{16}$ | 2.0616 | .3382 | $2\frac{3}{4}$ | 8.2467 | 5.4119 | $4\frac{1}{2}$ | 15.511 | 19.147 |
| $1\frac{5}{16}$ | 2.1598 | .3712 | $2\frac{1}{2}$ | 8.443 | 5.6723 | 5 | 15.708 | 19.635 |
| $1\frac{7}{16}$ | 2.258 | .4057 | $2\frac{3}{8}$ | 8.6394 | 5.9395 | $5\frac{1}{8}$ | 15.904 | 20.129 |
| $1\frac{9}{16}$ | 2.3562 | .4417 | $2\frac{1}{2}$ | 8.8357 | 6.2126 | $5\frac{1}{4}$ | 16.1 | 20.628 |
| $1\frac{11}{16}$ | 2.4543 | .4793 | $2\frac{5}{8}$ | 9.0321 | 6.4918 | $5\frac{1}{2}$ | 16.296 | 21.135 |
| $1\frac{13}{16}$ | 2.5525 | .5185 | $2\frac{3}{4}$ | 9.2284 | 6.7772 | $5\frac{1}{2}$ | 16.493 | 21.647 |
| $1\frac{15}{16}$ | 2.6507 | .5591 | 3 | 9.4248 | 7.0696 | $5\frac{1}{2}$ | 16.689 | 22.166 |
| $1\frac{1}{4}$ | 2.7489 | .6013 | $3\frac{1}{4}$ | 9.6211 | 7.3662 | $5\frac{1}{2}$ | 16.886 | 22.69 |
| $1\frac{3}{4}$ | 2.847 | .645 | $3\frac{1}{2}$ | 9.8175 | 7.6699 | $5\frac{1}{2}$ | 17.082 | 23.221 |
| $1\frac{5}{4}$ | 2.9452 | .6903 | $3\frac{3}{4}$ | 10.014 | 7.9798 | $5\frac{1}{2}$ | 17.278 | 23.758 |
| $1\frac{7}{4}$ | 3.0434 | .737 | $3\frac{1}{2}$ | 10.21 | 8.2957 | $5\frac{1}{2}$ | 17.474 | 24.301 |
| $1\frac{9}{4}$ | 3.1416 | .7854 | $3\frac{5}{4}$ | 10.406 | 8.618 | 5 | 17.671 | 24.85 |
| $1\frac{11}{4}$ | 3.2397 | .8366 | $3\frac{3}{4}$ | 10.602 | 8.9462 | $5\frac{1}{4}$ | 17.867 | 25.406 |
| $1\frac{13}{4}$ | 3.3382 | .894 | $3\frac{1}{2}$ | 10.799 | 9.2807 | $5\frac{1}{4}$ | 18.064 | 25.967 |
| $1\frac{15}{4}$ | 3.4366 | 1.1075 | $3\frac{1}{2}$ | 10.995 | 9.6211 | $5\frac{1}{4}$ | 18.261 | 26.535 |
| $1\frac{1}{2}$ | 3.535 | 1.2271 | $3\frac{3}{4}$ | 11.191 | 9.968 | $5\frac{1}{4}$ | 18.457 | 27.108 |
| $1\frac{1}{2}$ | 4.1233 | 1.353 | $3\frac{3}{4}$ | 11.388 | 10.32 | $5\frac{1}{4}$ | 18.653 | 27.688 |

AREAS OF CIRCLES ADVANCING BY 10THS

| Diam. | Areas. | | | | | | | | | | Diam. |
|-------|----------|---------|---------|---------|---------|---------|---------|---------|---------|---------|-------|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| 0 | 0 | 0.078 | 0.314 | 0.706 | 1.256 | 1.963 | 2.837 | 3.848 | 5.026 | 6.361 | 0 |
| 1 | 7.854 | 9.503 | 1.1303 | 1.3273 | 1.5393 | 1.7671 | 2.0106 | 2.2698 | 2.5446 | 2.8352 | 1 |
| 2 | 3.1416 | 3.4686 | 3.8013 | 4.1547 | 4.5239 | 4.9087 | 5.3093 | 5.7253 | 6.1575 | 6.6052 | 2 |
| 3 | 7.0636 | 7.5476 | 8.0424 | 8.5530 | 9.0792 | 9.6211 | 10.1787 | 10.7521 | 11.3411 | 11.9459 | 3 |
| 4 | 12.5664 | 13.2025 | 13.8544 | 14.5220 | 15.2053 | 15.9043 | 16.6190 | 17.3494 | 18.0951 | 18.8574 | 4 |
| 5 | 19.6350 | 20.4282 | 21.2372 | 22.0618 | 22.9022 | 23.7583 | 24.6301 | 25.5176 | 26.4208 | 27.3397 | 5 |
| 6 | 28.2744 | 29.2247 | 30.1907 | 31.1725 | 32.1699 | 33.1831 | 34.2120 | 35.2566 | 36.3168 | 37.3926 | 6 |
| 7 | 38.4946 | 39.5920 | 40.7151 | 41.8539 | 43.0085 | 44.1787 | 45.3647 | 46.5663 | 47.7837 | 49.0168 | 7 |
| 8 | 50.2676 | 51.5300 | 52.8102 | 54.1062 | 55.4174 | 56.7431 | 58.0831 | 59.4369 | 60.8218 | 62.2115 | 8 |
| 9 | 63.6174 | 65.0389 | 66.476 | 67.9292 | 69.3979 | 70.8823 | 72.3824 | 73.8982 | 75.4298 | 76.9770 | 9 |
| 10 | 78.5400 | 80.1186 | 81.7130 | 83.3230 | 84.9488 | 86.5903 | 88.2475 | 89.9204 | 91.6090 | 93.3133 | 10 |
| 11 | 95.0334 | 96.7691 | 98.5207 | 100.287 | 102.070 | 103.869 | 105.683 | 107.513 | 109.359 | 111.220 | 11 |
| 12 | 113.097 | 114.960 | 116.898 | 118.823 | 120.763 | 122.718 | 124.690 | 126.677 | 128.679 | 130.698 | 12 |
| 13 | 132.732 | 134.782 | 136.848 | 138.929 | 141.026 | 143.139 | 145.267 | 147.411 | 149.571 | 151.747 | 13 |
| 14 | 153.938 | 156.145 | 158.368 | 160.606 | 162.860 | 165.130 | 167.415 | 169.717 | 172.034 | 174.366 | 14 |
| 15 | 176.6715 | 179.079 | 181.458 | 183.854 | 186.265 | 188.692 | 191.134 | 193.593 | 196.067 | 198.556 | 15 |
| 16 | 201.062 | 203.583 | 206.120 | 208.672 | 211.241 | 213.825 | 216.424 | 219.040 | 221.671 | 224.318 | 16 |
| 17 | 226.980 | 229.658 | 232.352 | 235.062 | 237.787 | 240.528 | 243.285 | 246.057 | 248.846 | 251.650 | 17 |
| 18 | 254.469 | 257.304 | 260.125 | 263.022 | 265.905 | 268.803 | 271.716 | 274.646 | 277.591 | 280.552 | 18 |
| 19 | 283.529 | 286.521 | 289.529 | 292.553 | 295.598 | 298.648 | 301.719 | 304.805 | 307.908 | 311.026 | 19 |
| 20 | 314.160 | 317.309 | 320.474 | 323.655 | 326.852 | 330.064 | 333.292 | 336.536 | 339.795 | 343.070 | 20 |
| 21 | 346.361 | 349.667 | 352.990 | 356.328 | 359.681 | 363.051 | 366.436 | 369.837 | 373.253 | 376.685 | 21 |
| 22 | 380.133 | 383.597 | 387.076 | 390.571 | 394.082 | 397.608 | 401.150 | 404.708 | 408.282 | 411.871 | 22 |
| 23 | 415.478 | 419.097 | 422.733 | 426.385 | 430.053 | 433.737 | 437.436 | 441.151 | 444.881 | 448.628 | 23 |
| 24 | 453.390 | 456.168 | 459.961 | 463.770 | 467.595 | 471.437 | 475.292 | 479.164 | 483.052 | 486.955 | 24 |
| 25 | 490.875 | 494.809 | 498.760 | 502.726 | 506.708 | 510.706 | 514.719 | 518.748 | 522.793 | 526.854 | 25 |

METRIC EQUIVALENTS.

| | |
|---------------------------------------|--------------------------------------|
| 1 centimetre | = 0.3937 in. |
| 1 sq. centimetre | = 0.155 sq. in. |
| 1 cub. centimetre | = 0.0610 cub. in. |
| 1 kilogram | = 2.205 lb. |
| 1 kilogram-metre | = 86.82 in.-lb. |
| 1 centimetre per sec. | = 0.0328 ft. per sec. |
| 1 gram per sq. centimetre | = 0.0142 lb per sq. in. |
| 1 gram per cub. centimetre | = 62.42 lb per cub. ft. |
| Acceleration due to gravity | = 32.2 ft. per sec. |
| " " | = 981.4 centimetres per sec. |
| 1 in. | = 2.540 centimetres. |
| 1 sq. in. | = 6.451 sq. centimetres. |
| 1 cub. in. | = 16.38 cub. centimetres. |
| 1 lb. (avoirdupois) | = 453.6 kilogram. |
| 1 ft. per sec. | = 30.48 centimetres per sec. |
| 1 lb. per sq. in. | = 69.34 grams per sq. centimetre. |
| 1 lb. per cub. in. | = 27.616 grams per cub. centimetre. |
| 1 lb. per cub. ft. | = 0.016022 gram per cub. centimetre. |
| 1 in.-lb. | = 115.2 gram-centimetres. |

EQUIVALENT VALUES OF MILLIMETRES AND INCHES.

| Milli- metres. | Inches. | Milli- metres. | Inches. | Milli- metres. | Inches. | Milli- metres. | Inches. |
|-------------------|---------|-------------------|---------|-------------------|---------|-----------------------------|---------|
| 1 | ·0394 | 27 | 1·0630 | 53 | 2·0866 | 79 | 3·1103 |
| 2 | ·0787 | 28 | 1·1024 | 54 | 2·1260 | 80 | 3·1496 |
| 3 | ·1181 | 29 | 1·1417 | 55 | 2·1654 | 81 | 3·1890 |
| 4 | ·1575 | 30 | 1·1811 | 56 | 2·2047 | 82 | 3·2284 |
| 5 | ·1968 | 31 | 1·2205 | 57 | 2·2441 | 83 | 3·2677 |
| 6 | ·2362 | 32 | 1·2598 | 58 | 2·2835 | 84 | 3·3071 |
| 7 | ·2756 | 33 | 1·2992 | 59 | 2·3228 | 85 | 3·3465 |
| 8 | ·3150 | 34 | 1·3386 | 60 | 2·3622 | 86 | 3·3859 |
| 9 | ·3543 | 35 | 1·3780 | 61 | 2·4016 | 87 | 3·4252 |
| 10 | ·3937 | 36 | 1·4173 | 62 | 2·4410 | 88 | 3·4646 |
| 11 | ·4331 | 37 | 1·4567 | 63 | 2·4803 | 89 | 3·5040 |
| 12 | ·4724 | 38 | 1·4961 | 64 | 2·5197 | 90 | 3·5433 |
| 13 | ·5118 | 39 | 1·5354 | 65 | 2·5591 | 91 | 3·5827 |
| 14 | ·5512 | 40 | 1·5748 | 66 | 2·5984 | 92 | 3·6221 |
| 15 | ·5906 | 41 | 1·6142 | 67 | 2·6378 | 93 | 3·6614 |
| 16 | ·6299 | 42 | 1·6536 | 68 | 2·6772 | 94 | 3·7008 |
| 17 | ·6693 | 43 | 1·6929 | 69 | 2·7166 | 95 | 3·7402 |
| 18 | ·7087 | 44 | 1·7323 | 70 | 2·7559 | 96 | 3·7796 |
| 19 | ·7480 | 45 | 1·7717 | 71 | 2·7953 | 97 | 3·8189 |
| 20 | ·7874 | 46 | 1·8110 | 72 | 2·8347 | 98 | 3·8583 |
| 21 | ·8268 | 47 | 1·8504 | 73 | 2·8740 | 99 | 3·8977 |
| 22 | ·8661 | 48 | 1·8898 | 74 | 2·9134 | 100 | 3·9370 |
| 23 | ·9055 | 49 | 1·9291 | 75 | 2·9528 | | |
| 24 | ·9449 | 50 | 1·9685 | 76 | 2·9922 | | |
| 25 | ·9843 | 51 | 2·0079 | 77 | 3·0315 | (100 mm. = 1 decimetre.) | |
| 26 | 1·0236 | 52 | 2·0473 | 78 | 3·0709 | | |

KILOGRAMMES IN POUNDS.

| Kilos. | Pounds. | Kilos. | Pounds. | Kilos. | Pounds. | Kilos. | Pounds. |
|--------|---------|--------|---------|--------|---------|--------|---------|
| 1 | 2.205 | 26 | 57.320 | 51 | 112.436 | 76 | 167.551 |
| 2 | 4.409 | 27 | 59.525 | 52 | 114.640 | 77 | 169.756 |
| 3 | 6.614 | 28 | 61.729 | 53 | 116.845 | 78 | 171.960 |
| 4 | 8.818 | 29 | 63.934 | 54 | 119.049 | 79 | 174.165 |
| 5 | 11.023 | 30 | 66.139 | 55 | 121.254 | 80 | 176.370 |
| 6 | 13.228 | 31 | 68.343 | 56 | 123.459 | 81 | 178.574 |
| 7 | 15.432 | 32 | 70.548 | 57 | 125.663 | 82 | 180.779 |
| 8 | 17.637 | 33 | 72.752 | 58 | 127.868 | 83 | 182.983 |
| 9 | 19.842 | 34 | 74.957 | 59 | 130.073 | 84 | 185.118 |
| 10 | 22.046 | 35 | 77.162 | 60 | 132.277 | 85 | 187.393 |
| 11 | 24.251 | 36 | 79.366 | 61 | 134.482 | 86 | 189.597 |
| 12 | 26.455 | 37 | 81.571 | 62 | 136.486 | 87 | 191.802 |
| 13 | 28.660 | 38 | 83.776 | 63 | 138.891 | 88 | 194.010 |
| 14 | 30.865 | 39 | 85.980 | 64 | 141.096 | 89 | 196.211 |
| 15 | 33.069 | 40 | 88.185 | 65 | 143.300 | 90 | 198.416 |
| 16 | 35.274 | 41 | 90.389 | 66 | 145.505 | 91 | 200.620 |
| 17 | 37.479 | 42 | 92.594 | 67 | 147.710 | 92 | 202.825 |
| 18 | 39.683 | 43 | 94.799 | 68 | 149.914 | 93 | 205.030 |
| 19 | 41.888 | 44 | 97.003 | 69 | 152.119 | 94 | 207.234 |
| 20 | 44.092 | 45 | 99.208 | 70 | 154.323 | 95 | 209.439 |
| 21 | 46.297 | 46 | 101.413 | 71 | 156.528 | 96 | 211.644 |
| 22 | 48.502 | 47 | 103.617 | 72 | 158.733 | 97 | 213.848 |
| 23 | 50.706 | 48 | 105.822 | 73 | 160.937 | 98 | 216.053 |
| 24 | 52.911 | 49 | 108.026 | 74 | 163.142 | 99 | 218.275 |
| 25 | 55.115 | 50 | 110.231 | 75 | 165.347 | 100 | 220.462 |

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